1. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

   (a) $A = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$

   Answer. Since $\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -5 & -7 \\ 0 & 1 & 2 & -6 \end{bmatrix}$, we have $x_1 = 5x_3 + 7x_4$, $x_2 = -2x_3 + 6x_4$, with x_3 and x_4 free. In parametric form, the solution set is

$$\left\{ s \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

   (b) $A = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   Answer. Since $\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 5 & 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we have $x_1 = -5x_2 - 8x_4 - x_5$, $x_3 = 7x_4 - 4x_5$, $x_6 = 0$, with x_2, x_4, and x_5 free. In parametric form, the solution set is

$$\left\{ x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} : x_2, x_4, x_5 \in \mathbb{R} \right\}$$

2. Give the parametric equation of the line through $\mathbf{a} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ parallel to $\mathbf{b} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$.

   Answer. $\begin{bmatrix} 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} -7 \\ 8 \end{bmatrix}$

3. Give the parametric equation of the line through both $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$.

   Answer. $\begin{bmatrix} 0 \\ -4 \end{bmatrix} + t \left(\begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right)$

4. Mark each of the following statements as True or False:

   (a) The solution set to an equation $A\mathbf{x} = \mathbf{b}$ always contains the origin. **(T)**

   (b) The solution set for the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable. **(F)**

(c) The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through $\mathbf{v}$ that is parallel to $\mathbf{p}$. **(F)**

5. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{v}$. Prove that $\mathbf{v}$ is the only solution to $A\mathbf{x} = \mathbf{b}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Proof. Theorem 6 states that the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form of the form $\mathbf{w} = \boldsymbol{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is a solution to the corresponding homogeneous equation. So if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two different solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{w}_1 - \mathbf{p}$ and $\mathbf{w}_2 - \mathbf{p}$ are two different solutions to the equation $A\mathbf{x} = \mathbf{0}$. Similarly, if $\mathbf{v}_{h_1}$ and $\mathbf{v}_{h_2}$ are two different solutions to the equation $A\mathbf{x} = \mathbf{0}$, then $\boldsymbol{p} + \mathbf{v}_{h_1}$ and $\boldsymbol{p} + \mathbf{v}_{h_2}$ are two different solutions to $A\mathbf{x} = \mathbf{b}$. So one equation has only one solution if and only if the other equation has only one solution.