- You should know how work with a **system of equations**, a **vector equation**, and a **matrix equation**, interchangeably.
  - *You should be able to solve a linear system of equations like* $\begin{cases} 3x + 2y - z = 0 \\ x - 2y + 3z = 1 \\ x + y + z = 2 \end{cases}$
  - *You should be able to solve a vector equation like* $x \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
  - *You should be able to solve a matrix equation like* $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- In each case above, should be able to write the solution set using **free variables** or in **parametric form**.
- When solving $A\mathbf{x} = \mathbf{b}$, we use the RREF of the augmented matrix $[A \ \ ]$. In the RREF form, you should know that an inconsistent system must have $[0 \ \ 0 \ \ 0 \ldots 1]$ in its bottom row, and a consistent system either has a unique solution or else it has free variables, which give infinitely many solutions.
- *Be example smart! Be able to come up with your own examples of matrix equations with no solutions, examples with one solution, and examples with infinitely many solutions. Write your example also as a vector equation and as a system of linear equations.*
- You should know how to reason about **homogeneous** equations given only RREF information about the (non-augmented) A matrix.
- You should know how to perform addition and scalar multiplication for vectors.
- You should know how to multiply a matrix times a vector.
- You should know what a **linear combination** of a set of vectors is. You should know what the **span** of a set of vectors is. You should be able to solve problems about linear combination and span.
  - *If possible, write* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *as a linear combination of vectors* $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - *Determine if* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *is in SPAN* $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.
  - *Determine if the columns of* $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ *span all of* $\mathbb{R}^3$.
- You should know the official definitions of **linear dependence** and **linear independence**. You should be able to use linear algebra to check if a given set of vectors is independent or not.
  - *Is the set of vectors* $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ *linearly independent?*

– *For what value of h is the set of columns of the matrix* $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & h \end{bmatrix}$ *linearly dependent?*

- You should know the defining properties of linear transformations, and you should be able to use linear algebra to check whether a transformation is **one-to-one** and whether it is **onto**.

- **Examples!** You should know examples that illustrate the definitions.

  – *Give an example of a homogeneous matrix equation with infinitely many solutions.*

  – *Give an example of a set of three vectors in* $\mathbb{R}^3$ *that are linearly dependent.*

  – *Give an example of a set of two vectors in* $\mathbb{R}^2$ *that are linearly independent.*

  – *Give an example of a linear transformation that is one-to-one but not onto.*

- **Connections!** We have talked about several different problems, and ultimately the answer has something to do with the RREF form of an augmented matrix. For each of the following, see if you can write a question about RREF for a particular augmented matrix. Be sure to give the specific augmented matrix, but don't bother answering the question.

  – *Can* $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *be written as a linear combination of the vectors in the set* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ *?*

  – *Is the linear transformation given by* $T(x, y) = (x, x + y, x + y + z)$ *a one-to-one function?*

  – *Is* $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *in the set Span* $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ *?*

  – *Is the set* $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ *linearly independent?*

- **Counterexamples!** For an "if-then" statement, a counterexample is an example that satisfies the "if" part but not the "then" part. All of the statements below are FALSE. In each case, give an example that satisfies the hypothesis but not the conclusion.

  – *If* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ *are vectors in* $\mathbb{R}^4$ *and* $\mathbf{v}_3$ *is* not *a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$*, then* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ *is linearly independent.*

  – *If* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ *are vectors in* $\mathbb{R}^4$ *and* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent, then* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ *is linearly independent.*

  – *If a linear transformation T is one-to-one, then T must also be onto.*

  – *If If* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *are vectors in* $\mathbb{R}^4$ *and* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent, then Span* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^4$.

  – *Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. If the matrix equation $A\mathbf{x} = \mathbf{0}$ is consistent, then so is the matrix equation $A\mathbf{x} = \mathbf{b}$.*

  – *Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. If the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*