# Chapter 6 Notes

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# Definition of Inner Product

- For vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, we define the *inner product* (a.k.a., dot product) to be

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- Note that the inner product of $\mathbf{u}$ and $\mathbf{v}$ is closely related to the matrix product $\mathbf{u}^T\mathbf{v}$.

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- $\mathbf{u} \cdot (c\mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

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- Using this idea, we can define the *distance* between vectors $\mathbf{u}$ and $\mathbf{v}$ as $||\mathbf{u}-\mathbf{v}||$.

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- For orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$, the Pythogorean Theorem holds:
$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

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- For any matrix $A$, $(\mathrm{Row}(A))^{\perp} = \mathrm{Null}(A)$.

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- An *orthogonal basis* for a subspace $W$ is an orthogonal set that is also a basis for $W$.

# Benefits of Orthogonal Bases

**Example:** Write the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$.

# Orthogonal Projection

**Definition.** Suppose $L$ is the line through the origin formed by Span$\{\mathbf{u}\}$ and $\mathbf{y}$ is some other vector in the same space. The *orthogonal projection of* $\mathbf{y}$ *onto* $L$ (equivalently, onto $\mathbf{u}$) is given by

$$\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

**Side effect.** The distance from $\mathbf{y}$ to $L$ is just $||\mathbf{y} - \hat{\mathbf{y}}||$.

# The Orthogonal Decomposition Theorem

**Theorem.** Let $W$ be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$.
In fact, if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for $W$, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_p + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$