Finite Rings

MAT 320

Recall that you were asked to sketch a proof showing that in $\mathbb{Z}_n$ every nonzero element is either a unit or a zero divisor. The first problem is to show that this is true for any finite commutative ring with identity.

1. Let R be a finite commutative ring with identity. Show that every nonzero element of R is either a unit or a zero divisor.

   Here's a start. Let $0_R \neq a \in R$. If a is a zero divisor, then we are done. Suppose that a is not a zero divisor. We will show that a is a unit.

Our next goal is to solve problems 4 and 6. We need to solve problem 2 first.

2. Let R be a finite commutative ring. Show that if there is a nonzero element $a \in R$ that is not a zero divisor, then R has an identity.

   (a) First let $R = \{r_1, r_2, \ldots, r_t\}$ and show that for each r_i there is an integer n_i such that $a^{n_i} r_i = r_i$.

   (b) Let $n = n_1 n_2 \cdots n_t$ and show that $a^n = 1_R$.

3. How is problem 4 different from problem 1?

4. Let R be a finite commutative ring. Show that every nonzero element of R is either a unit or a zero divisor. (Hint: Use problems 1 and 2.)

5. How is problem 6 different from Theorem 3.11 which states that every finite integral domain is a field?

6. Let R be a finite commutative ring with no zero divisors. Show that R is a field. (Hint: Use problem 2 and Theorem 3.11.)

Notice that in problem 2 a power of a is equal to 1_R. This is true for any unit in finite commutative ring with identity. This is the content of problem 8. First we need to solve problem 7.

7. Let R be a ring with identity. Let u_1, u_2, $\ldots$, u_q be units in R and show that the product $u_1 u_2 \cdots u_q$ is a unit in R.

8. Let q be the number of units in a finite commutative ring with identity. Show that if u is a unit, then $u^q = 1_R$.

The next problem is not restricted to finite rings, but it is on this worksheet anyway.

9. Let R be a commutative ring. Suppose that a is an element of R such that every element of R (including a) can be written as a product of a and some other element in R. Show that R has an identity.