Rings with 4 Elements
MAT 320

The following five rings have 4 elements and have the same addition table as $\mathbb{Z}_4$:

**R1.** $\mathbb{Z}_4$ with the usual multiplication.

**R2.** $\mathbb{Z}_4$ with multiplication $ab = 0$ for all $a, b \in \mathbb{Z}_4$.

**R3.** $\{0, 2, 4, 6\}$ as a subring of $\mathbb{Z}_8$.

**R4.** $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \right\}$ as a subring of $M(\mathbb{Z}_4)$.

**R5.** $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \right\}$ as a subring of $M(\mathbb{Z}_4)$.

1. Using what we know about properties preserved by isomorphisms, classify these rings up to isomorphism. In other words, decide which rings are isomorphic.

2. For each class, determine if

   (a) the rings have a unit
   (b) how many zero divisors they have
   (c) how many units they have
The following five rings have 4 elements and have the same addition table as 
\( \mathbb{Z}_2 \times \mathbb{Z}_2 \):

**R6.** \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with the usual multiplication.

**R7.** \( \mathbb{Z}_2[i] = \{a + bi \mid a, b \in \mathbb{Z}_2\} \) (notice that \( i^2 = -1 = 1 \)).

**R8.** \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \) as a subring of \( M(\mathbb{Z}_2) \).

**R9.** \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \) as a subring of \( M(\mathbb{Z}_2) \).

**R10.** \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \) as a subring of \( M(\mathbb{Z}_4) \).

3. Which rings are fields?

4. Which rings are not commutative?

5. Using what we know about properties preserved by isomorphisms, classify these rings up to isomorphism. In other words, decide which rings are isomorphic.

6. For each class, determine if

   (a) the rings have a unit
   (b) how many zero divisors they have
   (c) how many units they have

This next question is separate from the others.

7. Determine whether the two subrings

   \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \)

   and

   \( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\} \)

   of \( M(\mathbb{Z}_2) \) are isomorphic. Notice that these rings are not commutative.