Conway’s Rational Tangles

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
May 30, 2013

1 Introduction

I watched Professor John Conway perform this demonstration a number of years ago. He calls it “Rational Tangles” and there is plenty of information about it on the internet. Since then I have used it myself in classrooms of students of middle school age and older. The underlying mathematics is very interesting, but it is not necessary that the students understand all the mathematics for the demonstration to be educational. In fact, some of the mathematics I do not understand.

This document is intended for teachers or anyone else who would like to do this demonstration and includes some pedagogical advice and tricks I have used to make the demonstration run smoothly.

We begin with four people standing at the corners of a rectangle, each holding one end of two ropes. Then, without ever letting go of their rope, they perform some sequence of “dance figures” where they exchange places in various ways and as a result, the ropes become tangled. What is amazing is that we can assign a “number” to each of the tangled states which is modified in a precise way by each of the dance figures. The initial (completely untangled) state is assigned the number zero and even though the tangle may become extremely complicated, by performing the proper sequence of figures that tangle’s number can be reduced to zero at which point the ropes will be completely untangled.

The operations corresponding to the dance figures force the students to practice their arithmetic operations on positive and negative rational numbers, so a wonderful side effect of this demonstration is that the students are tricked into drilling their arithmetic facts even though they think that all they’re doing is trying to get a pair of ropes untangled.

For students whose arithmetic with fractions is a little weak, that may be all that they get out of it, but there is also an opportunity for advanced students to look at far more interesting mathematics.

2 Getting Started

To demonstrate the trick, you need four students and two lengths of rope that are about 10 feet long. Thicker rope is better because it is easier for the rest of the class to see the knot structure and it is harder to accidentally pull into tight knots that are difficult to work with. If the ropes are of two different colors, the tangle structure is even easier to see. It is also nice to have a few plastic shopping bags.

Get four volunteers to stand at the corners of a rectangle at the front of the class with each student holding one end of a rope. In the initial configuration, the two ropes are parallel to each other and parallel to the front row of seats in the classroom. In Figure 1, the top pair of parallel lines represents the two ropes, and the small circles at the ends with the letters "A". "B", "C" and "D" represent the four students. If you imagine that you are looking down on the students from the ceiling, the rest of the class is seated above the entire figure on the page. Student "A" faces student "C" and student "B" faces student "D".

Make sure that each student has a solid grip on the rope, perhaps wrapping the end once around their wrist so that it is not accidentally dropped. During the trick, no student should ever let go of his or her end of the rope. Don't let the kids start jerking on the rope, since if one end comes loose, it is very easy to lose track of exactly how the ropes were tangled, and if this occurs, the trick will fail, and the class will lose interest rapidly. Also, although the trick works for arbitrarily complex tangles, be sure to work with simple ones at first since there is much less chance of an error.

You can explain to the kids that they are going to do something like a square dance where the four students perform one of three dance figures (but if you've got shy students, it's probably best to wait until the volunteers are at the front of the room before you mention the word "dance"). Also explain that the initial configuration with the parallel, untangled ropes will be assigned the number zero which will indicate "completely untangled", and that the performance of each dance figure will affect that number in a fixed way. Finally, tell the rest of the kids in the class to pay attention, since you'll swap out sets of kids from time to time so that many more of them can be part of the action.

The only thing that matters is the configuration of the ropes: which student is in which position does not affect the number assigned to a particular tangle. The colors of the ropes don't make any difference, either. For example, the tangle will be in the "zero," or completely untangled position whether the red rope or the green rope is nearest the students in the class.

3 The Three Basic Dance Figures

Conway calls the two main dance figures "Twist 'em up" and "Turn 'em around". The unfortunate thing about this choice is that they both begin with the letter "T". If you're trying to analyze the results of various sequences, these names do not provide an easy shorthand. Here I will use "Twist" and "Rotate", since then you can write something like "$TTRTR$" to indicate that sequence of 5 figures in the dance (in this case, two twists, followed by a rotate, then a twist, and finally, another rotate). In what follows, I will use the names **Twist** and **Rotate**, and "T" and "R" as shorthand, especially when I need to refer to a sequence of moves. (In fact, we will see that when a sequence is repeated, we can also use an exponential notation if the kids are familiar with it. For example, the sequence $TTTTRTTTR$ could be written using the shorthand T^4RT^3R.)

Later in this article we will get even a little more sloppy and say things like "apply a TTR" as a shorthand for "apply a **Twist**, then another **Twist** and finally a **Rotate**".

When explaining the move "**Twist**", make sure that all four students pay attention, since although only two of them perform any particular **Twist**, they may be arranged differently later in the dance and will have to do it when they are in those positions.

To perform the **Twist** dance figure, the two students on the left (from the point of view of the students in the class) change places, with the student initially in the rear lifting his or her rope and the student in front stepping under it. As it is labeled in Figure 1, students A and B swap places, where student B lifts their rope and A steps under it to the rear. In Figure 1 the results of performing zero, one, two and three of the **Twist** dance figures (or, in shorthand, performing T^0, T^1, T^2 and T^3) from the initial (zero) configuration are shown from top to bottom. Notice that from the points of view of *all* of the students holding the ropes, after starting

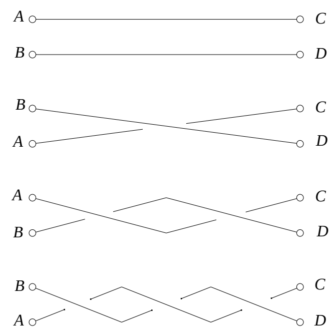

Figure 1: Twisting

from zero and performing just **Twist** moves, the ropes will appear to twist away from them in a clockwise direction. Notice that with each **Twist**, the positions of students A and B are swapped. Demonstrate this to the class. Near the end of this document, in Figure 3 are some photos of tangles made with real rope that are probably easier to visualize than those in Figure 1.

Each time a **Twist** move is made, the number associated with the tangle is increased by 1, so in Figure 1 the four tangles from top to bottom are represented by the numbers 0, 1, 2 and 3. If the tangle's number is a pure integer like this, then the integer represents the number of half-twists in the rope.

Conway's dance figures for manipulating tangles do not admit an **UnTwist** figure (which would exactly undo a **Twist**), but if there were one, it would be easy to do: the same two people on the left change places, but this time the person in front raises the rope and the person behind steps under it. Such an **UnTwist** dance figure would subtract 1 from the tangle's associated number. This is a very obvious concept, and if **Twist** and **UnTwist** were the only two legal moves, it's clear that starting from zero, any positive or negative integer could be obtained, and if you knew that number, the ropes could be untangled by performing that number of **UnTwist**s or **Twist**s, depending on whether the number were positive or negative.

In fact, a dance that only had **Twist** and **UnTwist** commands would be very simple to model, mathematically. Every reachable tangle would have a number that is a positive or negative integer, or zero. Every **Twist** adds one and every **UnTwist** subtracts one. Of course the "tangles" will be pretty simple: just spirals of rope twisted one way or the other. Notice also that if you know that the current number in such a model happens to be 5, there is no way to know exactly how you got there: any combinations of **Twist** and **UnTwist** that contains 5 more **Twist**'s than **UnTwist**'s will yield a 5.

The second dance figure, **Display**, does nothing to the tangle; it is simply to display the condition of the ropes and tangle to the rest of the class. To do a **Display**, the two people farthest from the class raise their ropes and the two in front lower them so the tangle is displayed in an unobstructed way. Conway usually also required that everyone in the class cheer and clap when a **Display** dance figure was performed. The cheering is a good idea, as it gives the class something fun to do.

To perform the third dance figure, **Rotate**, each student moves one position clockwise, when viewed from above. In Figure 1 if we began from the top arrangment in the figure, a **Rotate** would move A to C's position, B to A's position, D to B's position and C to D's position. If you were to **Rotate** four times in a row, each student would wind up exactly where they started. Demonstrate to the class that at least when there are only twists in the rope (and in fact it will always be true) that two **Rotate** dance figures will return the ropes back to where they started, even though the students will be on the opposite sides. Perhaps this can be made clear by reminding the students that the number 3, for example, represents 3 clockwise half-twists of the ropes from the point of view of any of the students. As they turn around, nothing is going to change the clockwise orientation, so after the two pairs on the ends have swapped places, they still see three clockwise half-twists, so rope configuration is unchanged.

This observation indicates that the operation on the number associated with the tangle has to bring it back to where it started if you apply the **Rotate** operation twice. Depending on the sophistication of the class, you might use functional notation as follows:

Let x be the number associated with the current tangle. If we apply a **Twist**, we'll use the function $t(x) = x + 1$ to indicate what a **Twist** does to the current number. At this point, we don't know what the **Rotate** number $r(x)$ does, but we do know two things. Clearly, $r(r(r(r(x)))) = x$, since rotating everyone completely around the square obviously leaves everything completely unchanged, no matter what the tangle. We also know for sure that if the tangle consists only of twists, then $r(r(x)) = x$.

Warning: This functional notation may be confusing, since the functions have to be written in the opposite order that the dance figures are performed. For example, if we start from a tangle whose associated number is x and do a **Twist** followed by a **Rotate**, we've been using the notation "TR" to indicate that: "twist, then rotate". But to figure out what the resulting corresponding number should be, the **Twist** will turn x into $t(x)$ and the **Rotate** function will operate on $t(x)$ to produce $r(t(x))$. It's easy to see how this reversal will always occur, so that something like "$TTRTTTRT = T^2RT^3RT$" will convert an initial number x to:

$$t(r(t(t(t(r(t(t(x)))))))).$$

So the bottom line is that unless you've got a sophisticated audience, it's probably a good idea to avoid the functional notation.

4 What Mathematical Operation Represents Rotate?

At this point we still don't know exactly how **Rotate** should affect the tangle's associated number. All we know (or at least suspect) is that applying **Rotate** twice brings us back to where we started. In other words, $r(r(x)) = x$,

For another clue about how **Rotate** should affect the number (or alternatively, the form of $r(x)$), have the students do this: Start from the ropes in a "zero" tangle. Do one **Twist** (so the number is now 1). Next do a **Rotate**. Finally do another **Twist**, and they will find that this brings the ropes back to the untangled state; namely, zero. This means that after the **Rotate**, the number must have been -1, since adding 1 to it brings us back to a 0 configuration. So **Rotate** changes a 1 to a -1. (Alternatively, using the functional notations we could write: $r(1) = -1$.)

The class will then probably make the reasonable (but wrong) guess that a **Rotate** dance figure multiples the number by -1. Sometimes they even guess that it adds 2. You can convince them that adding 2 is clearly wrong, since doing it twice should return to the original number and adding 2 twice will add 4 to the original number. The conjecture that **Rotate** multiplies the number by -1 (or functionally, that $r(x) = -x$) makes sense, since multiplying by -1 twice returns to the initial number. But this is easy to test: Start from 0, do two **Twist**s (which will convert the tangle's number to 2) followed by a **Rotate**. If **Rotate** multiplies by -1, then the ropes should then be in the -2 state, and two **Twist**s should add 2 to the -2, returning the tangle to the initial state. Try it, and see that this does not happen.

Depending on the sophistication of the class, you can either tell them the answer and go on, or try to lead them to it by considering other operations that turn 1 into -1 but not 2 into -2, yet when repeated twice bring every number back to itself.

I have found that it is useful to make boards out of wood or something that are shaped like those illustrated in Figure 2. Cut slots in the four corners that are a tiny bit narrower than the cord that you use to represent the ropes. Cut pieces of cord that are perhaps 2 times the length of the diagonal of the board and tie knots at both ends. Students can then use this board to run experiments with various combinations of **Twist** and **Rotate**. I've found that one board plus a pair of cords for every four or five students is sufficient. The cords can be slipped into the notches on the corners of the board and they will stay in place due to the knot on the end and the fact that there's some friction because the cord is a tiny bit fatter than the slot. With such a board even a single person can easily run experiments without the need for four hands.

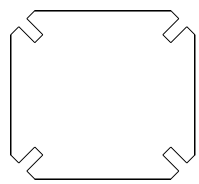

Figure 2: Test Board

Making a **Twist** simply involves swapping the cords on one side of the board and a **Rotate** is accomplished by rotating the entire board.

If you've got the boards, send the students in the front back to their seats and pass out borads to the groups of kids.

With the students divided into groups, ask them to try to find a combination of **Twist**'s and **Rotate**'s that will return a 2 (in other words, two **Twist**'s from zero). If they get the 2-**Twist** solution, have them look for a way to undo 3, 4 or 5 initial **Twist**'s. Have them write down the patterns they find in an organized way so that they can look for patterns. (Include the inverse of T, since that may make the pattern easier to see.) The nice thing about this is that even if there's a slow group at finding the 2-**Twist** solution, by the time the fastest group has the 4- or 5-**Twist** solution (or even the general pattern), the slowest group can almost always make some progress.

For reference, here are the sequences that undo the patterns listed above as well as the general formula, which some of the groups are likely to find:

Initial Position	Inverse
T	RT
$TT = T^2$	$RTRTT$
$TTT = T^3$	$RTRTTRTT$
$TTTT = T^4$	$RTRTTRTTRTT$
T^n	$RT(RTT)^{n-1}$

Get the class to see the pattern and then have them test the result using their boards on something more complex, like T^6.

Now we have a little more data; perhaps enough to help us figure out what mathematical operation we should assign to **Rotate**.

Here is one idea: First remind the class that, at least for pure twisting, the mirror image of a twist is represented by the negative of the number represented by the original pure twist. Do a TTR which produces a number x. If we can figure out what x is (as a number) then we will know what **Rotate** does to the number 2. Have the class take a good look at the ropes in this configuration. Follow that by a T (which produces $x + 1$) and notice that the result is just a mirror image of x. In the same way that -1 and 1 are mirror images, x and $x + 1$ are also. This means that another good guess might be that $-x = x + 1$, which we can solve to yield $x = -1/2$. If this is true, then we know that the **Rotate** command converts a 2 into $-1/2$.

If the class is a little bit sophisticated, here is another nice thing to do. Start with the usual zero configuration and do a single **Rotate**. Now do any number of **Twist** operations. The **Twist** operations have no effect: the ropes remain parallel to each other, but perpendicular to the front row of the class. This means that whatever number $r(0)$ happens to be (let's call it i, for now), this "number" i has the strange property that $i + 1 = i$. Now there aren't any normal numbers like that, but maybe one of the kids will come up with the idea that $\infty + 1 = \infty$, so perhaps $i = \infty$, whatever that means.

If that's the case, we have $r(r(x)) = x$, we have $r(1) = -1$, it seems likely that $r(2) = -1/2$ and we have $r(0) = \infty$ (and obviously, if we rotate the "∞" configuration we'll return to zero, so $r(\infty) = 0$). These may provide enough clues for a sophisticated class to determine the correct operation to associate with the **Rotate** dance figure. (Actually, an even better number to assign might be $-\infty$, but that may require even more sophistication on the part of the students.)

The correct answer is that **Rotate** takes the tangle represented by x and turns it into the tangle represented by $-1/x$. Thus, starting from zero, the sequence TTR leaves a tangle with value $-1/2$. You can check this by starting from zero, doing a TTR (which should leave $-1/2$), then doing a T (yielding a value of $-1/2 + 1 = +1/2$), then an R (yielding -2) and then a TT brings you back to zero. Have the class check that this works for the examples examined so far, and if they've discovered the ∞ idea, that it makes some sort of sense even then.

5 Getting to Zero

Get a new set of students in front of the class with the large set of ropes.

At this point you can begin to consider how an arbitrary tangling of the rope using the **Twist** and **Rotate** commands can be converted back to the untangled state again using only the **Twist** and **Rotate** commands. You might begin by having the kids suggest moves that tangle the rope a little bit: perhaps seven or eight steps, but carefully keeping track of the numbers. For now, make sure they start with at least two **Twist**s and mix in both **Twist**s and **Rotate**s after that.

If they try to do two **Rotate**s in a row, point out that although this is perfectly legal, the second **Rotate** undoes the first, so the two moves taken together achieve nothing.

What you can do is collect move suggestions from different kids, keeping track of the tangle's number until it is suitably complex. (Of course the meaning of "suitably" depends completely on the arithmetic abilities of the students in the class). Also, as your first example, stop them after a command has left the number positive. For example, the sequence $TTRTTTRT$, starting from zero, leaves a tangle with associated number $3/5$. This is a nice number since it doesn't have numerator or denominator that's too big, but it is complicated enough to be interesting. We'll use this example in what follows.

Tell the kids that their goal is to get the number $3/5$ down to zero using only **Rotate** and **Twist** commands. As a first hint, tell them that **Twist** will add 1 which will take the number even farther away from zero, so to make progress, probably the best possible command to start with is **Rotate**. We now have $-5/3$.

Maybe they can solve it from this point on, but if not, point out now that another **Rotate** will just undo the one they did, so the only reasonable next step is a **Twist**, yielding $-2/3$. At this point a **Rotate** will not put you back where you started, but it would yield a positive number, and that can't be good, since another **Rotate** is useless, and one or more **Twist** commands would take the number away from zero. Thus from $-2/3$, the only reasonable move is another **Twist**, yielding $1/3$.

Repeating the arguments above, we clearly need a **Rotate**, taking us to -3, and then three **Twist**s get back to zero. Go ahead and do this with the ropes and verify that indeed it does untangle the mess.

As the kids are working to undo the tangle represented by, say, $3/5$, I like to keep track of the calculation on the board with a drawing like this, where we add the arrows and new values as each operation is performed:

$$\frac{3}{5} \xrightarrow{R} -\frac{5}{3} \xrightarrow{T^2} \frac{1}{3} \xrightarrow{R} -3 \xrightarrow{T^3} 0.$$

6 Next Steps

Depending on the amount of time available and the sophistication of the audience you can then cover some of the topics in the following sections. But whatever you do, leave enough time for the following:

Make another tangle, a bit more involved than the ones you have considered up to this point, and once it's created, put the tangle into a bag as follows. Take a plastic bag and make two small holes in the corners opposite the opening. Take the ropes, one at a time, from the two kids on the left and feed them through the holes and back to the kid. Pull the bag opening over the tangle and tie the whole thing shut so that the tangle is completely enclosed in the bag.

Finally, carefully apply the steps that undo the tangle and when you're done, there will be a horrible snarl of ropes and plastic, which, if you've made no mistakes, should be equivalent to zero. To prove it, tear the plastic bag into pieces to extract it from the tangle, and then with a few tugs, the entire mess will appear to magically untangle itself!

Obviously, you need to be *very* careful with the arithmetic and to make sure that the kids undoing the tangle do exactly the correct calculations. I usually ask everyone in the audience to check the arithmetic and to check that the kids on the ends of the ropes do exactly what they're supposed to.

7 "Bad" Tangles

The following tangles are perfectly ok in a mathematical sense, but untangling them is a long process, and can be quite error-prone. For that reason, unless you've got masochistic tendencies, avoid tangles with numbers like $-1/n$, where n is a large integer. They are easy to produce: suppose you start from zero and do $TTTTTTTTR$: eight **Twist** dance figures followed by a **Rotate**. This will produce the number $-1/8$. It's a good exercise to try to untangle it using our method to see what happens. Here's what happens, shown as a series of steps:

Start	Operation	Result
$-1/8$	TRT	$-1/7$
$-1/7$	TRT	$-1/6$
$-1/6$	TRT	$-1/5$
$-1/5$	TRT	$-1/4$
$-1/4$	TRT	$-1/3$
$-1/3$	TRT	$-1/2$
$-1/2$	TRT	-1
-1	T	0

It requires 22 moves to return $-1/8$ to zero. Not only that, but the arithmetic is pretty boring: go though the details for a couple of the rows above to see what happens.

8 Discussion Topics

Here are a few ideas that may lead to interesting class discussions:

8.1 Infinity as a Tangle Number

Try starting with zero and do a single **Rotate**. This yields the nonsense value $-1/0$, but it's not a nonsense tangle. Another **Rotate** will bring it back to zero, and in fact, it sort of behaves like "infinity" in the sense that a **Twist** (try it) leaves it exactly the same. This sort of corresponds to the idea that adding 1 to ∞ leaves it unchanged. You may have discussed this earlier, depending on the sophistication of the class.

8.2 Proof of Convergence to Zero

Can you prove that the scheme outlined above will always eventually grind any initial fraction down to zero? Go through a few examples and see what is happening. Here is an example starting from $-5/17$:

$$-\frac{5}{17} \xrightarrow{T} \frac{12}{17} \xrightarrow{R} -\frac{17}{12} \xrightarrow{TT} \frac{7}{12} \xrightarrow{R} -\frac{12}{7} \xrightarrow{TT} \frac{2}{7} \xrightarrow{R} -\frac{7}{2} \xrightarrow{TTTT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0.$$

Note that after each **Rotate** command, the resulting negative fraction has a smaller denominator. Why is this? If the denominators always eventually get smaller, they must eventually get to 1. But when a denominator is 1 the fraction will be a negative integer, and if that integer happens to be $-n$, we know that n **Twist** commands (each adding 1) will reduce it to zero.

8.3 Relationship to the Greatest Common Divisor

If the students are a bit advanced, you can point out that the process of reducing the fraction down to zero is almost exactly the same as finding the greatest common divisor (the GCD) of the numerator and denominator. Since we begin with a fraction reduced to lowest terms, this will always get us down to 1 as the GCD.

Euclid's algorithm for calculating the GCD of two numbers works as follows. If the two numbers are m and n, and we suppose that $m > n$, we can write: $m = kn+l$, where $k \geq 1$ is an integer and $|l| < n$. Any number that divides m and n must divide l in the equation above, so we can conclude that $GCD(m, n) = GCD(n, l)$. The numbers in the right hand side are reduced, and the process can be repeated until one is a multiple of the other.

Here is an example: find the GCD of 4004 and 700:

$$\begin{aligned} 4004 &= 700 \times 5 + 504 \\ 700 &= 504 \times 1 + 196 \\ 504 &= 196 \times 2 + 112 \\ 196 &= 112 \times 1 + 84 \\ 112 &= 84 \times 1 + 28 \\ 84 &= 28 \times 3. \end{aligned}$$

The GCD of 4004 and 700 must divide 504 from the first line, so $GCD(4004, 700) = GCD(700, 504)$. The same process can be continued to obtain:

$$\begin{aligned} &GCD(4004, 700) = GCD(700, 504) = GCD(504, 196) \\ &\quad = GCD(196, 112) = GCD(112, 84) = GCD(84, 28). \end{aligned}$$

But 84 is an exact multiple of 28, so $GCD(84, 28) = 28$, and we can therefore conclude that

$$GCD(4004, 700) = 28$$

as well.

Note that there is no requirement that the numbers on the right hand sides of the sequence of reductions be positive. All that we require for convergence is that they be smaller in absolute value than the smaller of the two values for which you are trying to obtain the GCD. Also note that division can be achieved by repeated subtraction, and if you simply check to see if the subtraction yields a non-positive number, you know you have gone far enough. It's sort of like backing up your car until you hear breaking glass, but it works!

With all that in mind, let's find the GCD of 5 and 17 using this totally crude method:

$$\begin{aligned} 5 &= 17 \times 1 - 12 \\ 17 &= 12 \times 1 + 5 = 12 \times 2 - 7 \\ 12 &= 7 \times 1 + 5 = 7 \times 2 - 2 \\ 7 &= 2 \times 1 + 5 = 2 \times 2 + 3 = 2 \times 3 + 1 = 2 \times 4 - 1 \\ 2 &= 1 \times 1 + 1 = 1 \times 2 + 0. \end{aligned}$$

Note the similarity of this method to the one we used to obtain $GCD(4004, 700)$ above. But this time, rather than doing a division, we do repeated subtractions until the remainder is zero or negative. Then we use the (positive value of) the remainder in the next step. We finally discover that 1 divides 2 evenly, so $GCD(5, 17) = 1$. Now compare this sequence to the one that reduces the tangle value $-5/17$ to zero at the beginning of this section. You will see that the calculations are virtually identical.

8.4 What Tangle Numbers Are Possible?

Is it possible to start from zero and get to any (positive or negative) fraction? Have the students mess around for a while and see what fractions they can come up with. Also, set goals, such as, "Can you start from zero and get to -3?" If there is no progress, here is a giant hint:

$$\frac{3}{1} \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} -\frac{3}{2} \xrightarrow{TT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0$$

$$0 \xrightarrow{TT} \frac{2}{1} \xrightarrow{R} -\frac{1}{2} \xrightarrow{TT} \frac{3}{2} \xrightarrow{R} -\frac{2}{3} \xrightarrow{T} \frac{1}{3} \xrightarrow{R} -\frac{3}{1}.$$

If we start from 3 and work our way to zero using our standard methods, the sequence $RTRTTRTT$ does the trick. But now note that if we start from zero and use the reverse of the sequence above, namely: $TTRTTRTR$, we get to -3. Also, note that at every stage in the sequence, the same fractions are generated, except that they have opposite signs.2

8.5 Minimum Steps from 0

From the previous section, we know that we can get to any fraction i/j by using our algorithm to grind $-i/j$ to zero, and then reversing the order of the dance figures. Is this the minimum number of steps?

The tables below show the minimum number of steps to get to positive and negative fractions between $1/1$ and 7/7 (the second shows the steps to get fractions between $-1/1$ and $-7/7$). We use exponents to reduce the size of the strings in the sense that we might express the sequence $TTRTTRTTT$ as $T(TRT)^2T^2$. An X in a slot indicates that the fraction represented by that slot is not reduced to lowest terms. So for example, to look up the shortest sequence that will get you from zero to $-5/7$, we use the second table below (since the fraction is negative). We look in the column headed by -5 and the row headed by 7 and find the following: $T(TTR)^3$, which would expand to $TTTRTTRTTR$, and it is easy to verify that this, in fact, will generate $-5/7$. Check some other examples.

Similarly, if you look in the column headed by 6 and the row headed by 4, you find an X, since $6/4$ is not reduced to lowest terms: you should have been looking for $3/2$ in column 3, row 2.

Table 1: Positive fractions:

	1	2	3	4	5	6	7
1	T	T^2	T^3	T^4	T^5	T^6	T^7
2	T^2RT	X	T^2RT^2	X	T^2RT^3	X	T^2RT^4
3	$T(TRT)^2$	T^3RT	X	$T(TRT)^2T$	T^3RT^2	X	$T(TRT)^2T^2$
4	$T(TRT)^3$	X	T^4RT	X	$T(TRT)^3T$	X	T^4RT^2
5	$T(TRT)^4$	$T^2(TRT)^2$	T^2RT^3RT	T^5RT	X	$T(TRT)^4T$	$T^2(TRT)^2T$
6	$T(TRT)^5$	X	X	X	T^6RT	X	$T(TRT)^5T$
7	$T(TRT)^6$	$T^2(TRT)^3$	$T^3(TRT)^2$	$T(TRT)^2T^2RT$	T^2RT^4RT	T^7RT	X

Table 2: Negative fractions:

	-1	-2	-3	-4	-5	-6	-7
1	TR	$T(TRT)R$	$T(TRT)^2R$	$T(TRT)^3R$	$T(TRT)^4R$	$T(TRT)^5R$	$T(TRT)^6R$
2	T^2R	X	$T^2(TRT)R$	X	$T^2(TRT)^2R$	X	$T^2(TRT)^3R$
3	T^3R	T^2RT^2R	X	T^4RTR	T^2RT^3RTR	X	T^4RT^2RTR
4	T^4R	X	$(T^2R)^3$	X	T^5RTR	X	$(T^2R)^2T^3RTR$
5	T^5R	T^2RT^3R	T^3RT^2R	$T(TRT)^3TR$	X	T^6RTR	T^2RT^4RTR
6	T^6R	X	X	X	$(T^2R)^5$	X	T^7RTR
7	T^7R	T^2RT^4R	$(T^2R)^2T^3R$	T^4RT^2R	$T(TTR)^3$	$(TTR)^6$	X

There are some obvious patterns here, and an interesting exercise is both to look at the patterns and then to try to prove them. Here is a list of some of the obvious ones; perhaps there are others.

$$\begin{aligned} T^n &: 0 \longrightarrow n \\ T(TRT)^n &: 0 \longrightarrow 1/(n+1) \\ T^2RT^n &: 0 \longrightarrow (2n-1)/2 \\ T^2(TRT)^n &: 0 \longrightarrow 2/(2n+1) \\ T(TRT)^nR &: 0 \longrightarrow -(n+1) \end{aligned}$$

On the following page is a list of fractions that can be obtained, starting from zero, by applying various sequences of T and R. The data on that page may also be useful to generate conjectures about sequence patterns and the fractions resulting from them.

Some sequences do not yield patterns that are obvious at first. For example, consider the sequence: T^2RT, T^3RT^2, T^4RT^3, $\ldots$, in other words, what does $T^{n+1}RT^n$ represent? The first few values are $1/2$, $5/3$, $11/4$, $19/5$, $29/6$. The denominators go up by 1 each time and the numerators by $4, 6, 8, 10$. A little fooling around will yield the formula $(n(n+1)-1)/(n+1)$, for $n > 0$. Following this idea can lead to an investigation of how to find formulas to represent the numbers in some series.

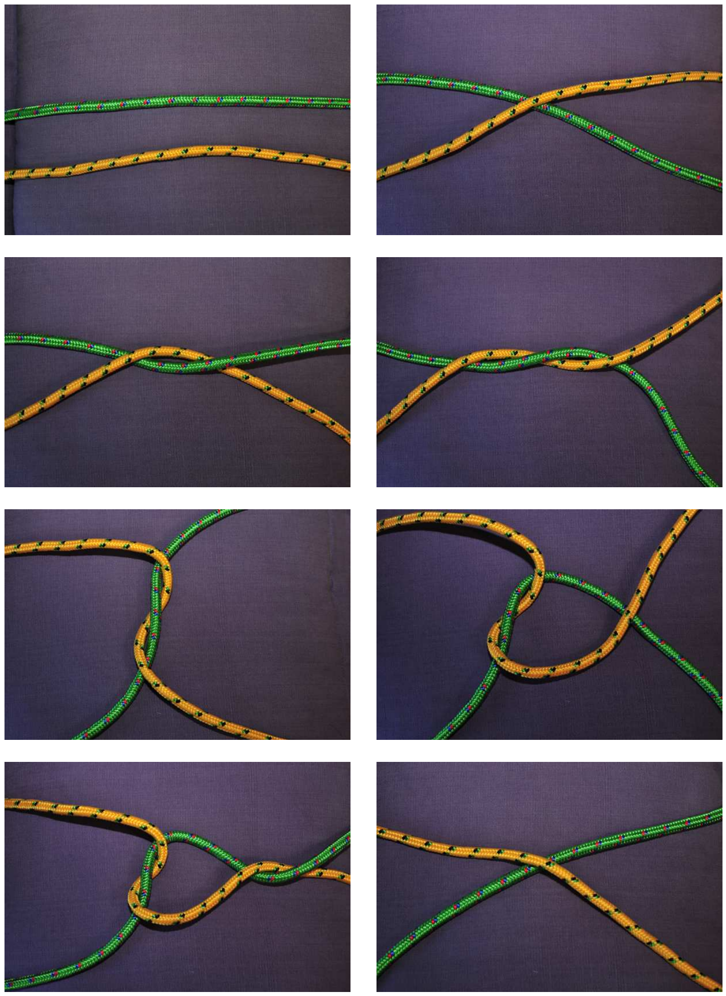

Figure 3: Tangles with real rope

In Figure 3 is illustrated a series of tangles as they appear with real rope. Reading from the top left, they represent the numbers:

$$0, 1, 2, 3, -1/3, -1/3 + 1 = 2/3, 2/3 + 1 = 5/3, \text{and } -1.$$

All but the final -1 are achieved from the previous tangle by a twist or a rotate. The final tangle, corresponding to -1, by performing rotate to the tangle in the upper right that represents 1.

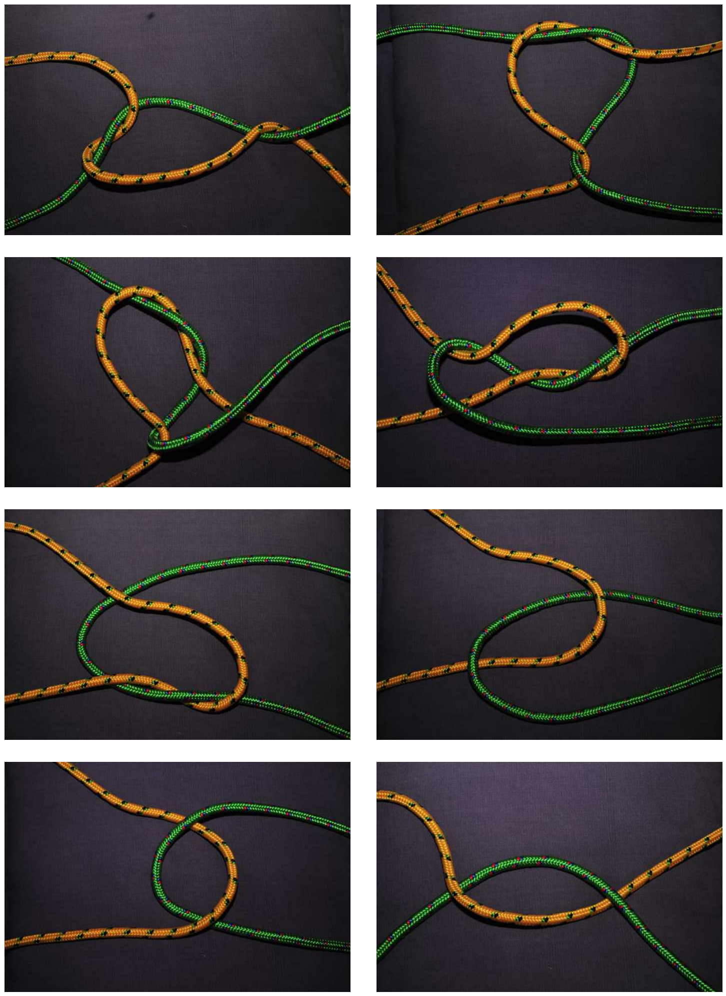

Figure 4: Turning $5/3$ back to 0.

Figure 4 illustrates the conversion of the tangle represented by $5/3$ back to zero. We begin with $5/3$ in the upper-left photo, which is the same as the $5/3$ displayed in Figure 3. It's easy to check that the sequence $RTRTTTR$ will convert that to -2 and each successive photo above shows the result after each of those 7 steps. It should be clear by looking at it that two more **Twist** dance figures will completely untangle the ropes in the final photo.

Here is a table listing the resulting fraction from various sequences of T and R:

R	∞	T^5RT^2R	$-5/9$	T^8RTR	$-8/7$	T^7RT^2RT	$6/13$
T	$1/1$	T^4RT^4	$15/4$	T^7RT^3	$20/7$	T^6RT^5	$29/6$
T^2	$2/1$	T^4RT^3R	$-4/11$	T^7RT^2R	$-7/13$	T^6RT^4R	$-6/23$
TR	$-1/1$	T^4RT^2RT	$3/7$	T^6RT^4	$23/6$	T^6RT^3RT	$11/17$
T^3	$3/1$	T^3RT^5	$14/3$	T^6RT^3R	$-6/17$	$T^6RT^2RT^2$	$16/11$
T^2R	$-1/2$	T^3RT^4R	$-3/11$	T^6RT^2RT	$5/11$	T^6RT^2RTR	$-11/5$
T^4	$4/1$	T^3RT^3RT	$5/8$	T^5RT^5	$24/5$	T^5RT^6	$29/5$
T^3R	$-1/3$	$T^3RT^2RT^2$	$7/5$	T^5RT^4R	$-5/19$	T^5RT^5R	$-5/24$
T^2RT	$1/2$	T^3RT^2RTR	$-5/2$	T^5RT^3RT	$9/14$	T^5RT^4RT	$14/19$
T^5	$5/1$	T^2RT^6	$11/2$	$T^5RT^2RT^2$	$13/9$	$T^5RT^3RT^2$	$23/14$
T^4R	$-1/4$	T^2RT^5R	$-2/9$	T^5RT^2RTR	$-9/4$	T^5RT^3RTR	$-14/9$
T^3RT	$2/3$	T^2RT^4RT	$5/7$	T^4RT^6	$23/4$	$T^5RT^2RT^3$	$22/9$
T^2RT^2	$3/2$	$T^2RT^3RT^2$	$8/5$	T^4RT^5R	$-4/19$	$T^5RT^2RT^2R$	$-9/13$
T^2RTR	$-2/1$	T^2RT^3RTR	$-5/3$	T^4RT^4RT	$11/15$	T^4RT^7	$27/4$
T^6	$6/1$	$T^2RT^2RT^3$	$7/3$	$T^4RT^3RT^2$	$18/11$	T^4RT^6R	$-4/23$
T^5R	$-1/5$	$T^2RT^2RT^2R$	$-3/4$	T^4RT^3RTR	$-11/7$	T^4RT^5RT	$15/19$
T^4RT	$3/4$	T^{10}	$10/1$	$T^4RT^2RT^3$	$17/7$	$T^4RT^4RT^2$	$26/15$
T^3RT^2	$5/3$	T^9R	$-1/9$	$T^4RT^2RT^2R$	$-7/10$	T^4RT^4RTR	$-15/11$
T^3RTR	$-3/2$	T^8RT	$7/8$	T^3RT^7	$20/3$	$T^4RT^3RT^3$	$29/11$
T^2RT^3	$5/2$	T^7RT^2	$13/7$	T^3RT^6R	$-3/17$	$T^4RT^3RT^2R$	$-11/18$
T^2RT^2R	$-2/3$	T^7RTR	$-7/6$	T^3RT^5RT	$11/14$	$T^4RT^2RT^4$	$24/7$
T^7	$7/1$	T^6RT^3	$17/6$	$T^3RT^4RT^2$	$19/11$	$T^4RT^2RT^3R$	$-7/17$
T^6R	$-1/6$	T^6RT^2R	$-6/11$	T^3RT^4RTR	$-11/8$	$T^4RT^2RT^2RT$	$3/10$
T^5RT	$4/5$	T^5RT^4	$19/5$	$T^3RT^3RT^3$	$21/8$	T^3RT^8	$23/3$
T^4RT^2	$7/4$	T^5RT^3R	$-5/14$	$T^3RT^3RT^2R$	$-8/13$	T^3RT^7R	$-3/20$
T^4RTR	$-4/3$	T^5RT^2RT	$4/9$	$T^3RT^2RT^4$	$17/5$	T^3RT^6RT	$14/17$
T^3RT^3	$8/3$	T^4RT^5	$19/4$	$T^3RT^2RT^3R$	$-5/12$	$T^3RT^5RT^2$	$25/14$
T^3RT^2R	$-3/5$	T^4RT^4R	$-4/15$	$T^3RT^2RT^2RT$	$2/7$	T^3RT^5RTR	$-14/11$
T^2RT^4	$7/2$	T^4RT^3RT	$7/11$	T^2RT^8	$15/2$	$T^3RT^4RT^3$	$30/11$
T^2RT^3R	$-2/5$	$T^4RT^2RT^2$	$10/7$	T^2RT^7R	$-2/13$	$T^3RT^4RT^2R$	$-11/19$
T^2RT^2RT	$1/3$	T^4RT^2RTR	$-7/3$	T^2RT^6RT	$9/11$	$T^3RT^3RT^4$	$29/8$
T^8	$8/1$	T^3RT^6	$17/3$	$T^2RT^5RT^2$	$16/9$	$T^3RT^3RT^3R$	$-8/21$
T^7R	$-1/7$	T^3RT^5R	$-3/14$	T^2RT^5RTR	$-9/7$	$T^3RT^3RT^2RT$	$5/13$
T^6RT	$5/6$	T^3RT^4RT	$8/11$	$T^2RT^4RT^3$	$19/7$	$T^3RT^2RT^5$	$22/5$
T^5RT^2	$9/5$	$T^3RT^3RT^2$	$13/8$	$T^2RT^4RT^2R$	$-7/12$	$T^3RT^2RT^4R$	$-5/17$
T^5RTR	$-5/4$	T^3RT^3RTR	$-8/5$	$T^2RT^3RT^4$	$18/5$	$T^3RT^2RT^3RT$	$7/12$
T^4RT^3	$11/4$	$T^3RT^2RT^3$	$12/5$	$T^2RT^3RT^3R$	$-5/13$	$T^3RT^2RT^2RT^2$	$9/7$
T^4RT^2R	$-4/7$	$T^3RT^2RT^2R$	$-5/7$	$T^2RT^3RT^2RT$	$3/8$	$T^3RT^2RT^2RTR$	$-7/2$
T^3RT^4	$11/3$	T^2RT^7	$13/2$	$T^2RT^2RT^5$	$13/3$	T^2RT^9	$17/2$
T^3RT^3R	$-3/8$	T^2RT^6R	$-2/11$	$T^2RT^2RT^4R$	$-3/10$	T^2RT^8R	$-2/15$
T^3RT^2RT	$2/5$	T^2RT^5RT	$7/9$	$T^2RT^2RT^3RT$	$4/7$	T^2RT^7RT	$11/13$
T^2RT^5	$9/2$	$T^2RT^4RT^2$	$12/7$	$T^2RT^2RT^2RT^2$	$5/4$	$T^2RT^6RT^2$	$20/11$
T^2RT^4R	$-2/7$	T^2RT^4RTR	$-7/5$	$T^2RT^2RT^2RTR$	$-4/1$	T^2RT^6RTR	$-11/9$
T^2RT^3RT	$3/5$	$T^2RT^3RT^3$	$13/5$	T^{12}	$12/1$	$T^2RT^5RT^3$	$25/9$
$T^2RT^2RT^2$	$4/3$	$T^2RT^3RT^2R$	$-5/8$	$T^{11}R$	$-1/11$	$T^2RT^5RT^2R$	$-9/16$
T^2RT^2RTR	$-3/1$	$T^2RT^2RT^4$	$10/3$	$T^{10}RT$	$9/10$	$T^2RT^4RT^4$	$26/7$
T^9	$9/1$	$T^2RT^2RT^3R$	$-3/7$	T^9RT^2	$17/9$	$T^2RT^4RT^3R$	$-7/19$
T^8R	$-1/8$	$T^2RT^2RT^2RT$	$1/4$	T^9RTR	$-9/8$	$T^2RT^4RT^2RT$	$5/12$
T^7RT	$6/7$	T^{11}	$11/1$	T^8RT^3	$23/8$	$T^2RT^3RT^5$	$23/5$
T^6RT^2	$11/6$	$T^{10}R$	$-1/10$	T^8RT^2R	$-8/15$	$T^2RT^3RT^4R$	$-5/18$
T^6RTR	$-6/5$	T^9RT	$8/9$	T^7RT^4	$27/7$	$T^2RT^3RT^3RT$	$8/13$
T^5RT^3	$14/5$	T^8RT^2	$15/8$	T^7RT^3R	$-7/20$	$T^2RT^3RT^2RT^2$	$11/8$

Here are the sequences required to return various fractions to zero, organized by denominator. The first twelve numerators that are relatively prime to the denominator are listed for each denominator.

1/1 : RT
2/1 : $RT\,RT^2$
3/1 : $RT\,RT^2\,RT^2$
4/1 : $RT\,RT^2\,RT^2\,RT^2$
5/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2$
6/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
7/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
8/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
9/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
10/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
11/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
12/1 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
1/2 : RT^2
3/2 : $RT\,RT^3$
5/2 : $RT\,RT^2\,RT^3$
7/2 : $RT\,RT^2\,RT^2\,RT^3$
9/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^3$
11/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
13/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
15/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
17/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
19/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
21/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
23/2 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
1/3 : RT^3
2/3 : $RT^2\,RT^2$
4/3 : $RT\,RT^4$
5/3 : $RT\,RT^3\,RT^2$
7/3 : $RT\,RT^2\,RT^4$
8/3 : $RT\,RT^2\,RT^3\,RT^2$
10/3 : $RT\,RT^2\,RT^2\,RT^4$
11/3 : $RT\,RT^2\,RT^2\,RT^3\,RT^2$
13/3 : $RT\,RT^2\,RT^2\,RT^2\,RT^4$
14/3 : $RT\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2$
16/3 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^4$
17/3 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2$
1/4 : RT^4
3/4 : $RT^2\,RT^2\,RT^2$
5/4 : $RT\,RT^5$
7/4 : $RT\,RT^3\,RT^2\,RT^2$
9/4 : $RT\,RT^2\,RT^5$
11/4 : $RT\,RT^2\,RT^3\,RT^2\,RT^2$
13/4 : $RT\,RT^2\,RT^2\,RT^5$
15/4 : $RT\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2$
17/4 : $RT\,RT^2\,RT^2\,RT^2\,RT^5$
19/4 : $RT\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2$
21/4 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^5$
23/4 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2$
1/5 : RT^5
2/5 : $RT^3\,RT^2$
3/5 : $RT^2\,RT^3$
4/5 : $RT^2\,RT^2\,RT^2\,RT^2$
6/5 : $RT\,RT^6$
7/5 : $RT\,RT^4\,RT^2$
8/5 : $RT\,RT^3\,RT^3$
9/5 : $RT\,RT^3\,RT^2\,RT^2\,RT^2$
11/5 : $RT\,RT^2\,RT^6$
12/5 : $RT\,RT^2\,RT^4\,RT^2$
13/5 : $RT\,RT^2\,RT^3\,RT^3$
14/5 : $RT\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2$
1/6 : RT^6
5/6 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
7/6 : $RT\,RT^7$
11/6 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2$
13/6 : $RT\,RT^2\,RT^7$
17/6 : $RT\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2$
19/6 : $RT\,RT^2\,RT^2\,RT^7$
23/6 : $RT\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2$
25/6 : $RT\,RT^2\,RT^2\,RT^2\,RT^7$
29/6 : $RT\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2$
31/6 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^7$
35/6 : $RT\,RT^2\,RT^2\,RT^2\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2$

1/7 : RT^7
2/7 : $RT^4\,RT^2$
3/7 : $RT^3\,RT^2\,RT^2$
4/7 : $RT^2\,RT^4$
5/7 : $RT^2\,RT^2\,RT^3$
6/7 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
8/7 : $RT\,RT^8$
9/7 : $RT\,RT^5\,RT^2$
10/7 : $RT\,RT^4\,RT^2\,RT^2$
11/7 : $RT\,RT^3\,RT^4$
12/7 : $RT\,RT^3\,RT^2\,RT^3$
13/7 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
1/8 : RT^8
3/8 : $RT^3\,RT^3$
5/8 : $RT^2\,RT^3\,RT^2$
7/8 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
9/8 : $RT\,RT^9$
11/8 : $RT\,RT^4\,RT^3$
13/8 : $RT\,RT^3\,RT^3\,RT^2$
15/8 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
17/8 : $RT\,RT^2\,RT^9$
19/8 : $RT\,RT^2\,RT^4\,RT^3$
21/8 : $RT\,RT^2\,RT^3\,RT^3\,RT^2$
23/8 : $RT\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
1/9 : RT^9
2/9 : $RT^5\,RT^2$
4/9 : $RT^3\,RT^2\,RT^2\,RT^2$
5/9 : $RT^2\,RT^5$
7/9 : $RT^2\,RT^2\,RT^2\,RT^3$
8/9 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
10/9 : $RT\,RT^{10}$
11/9 : $RT\,RT^6\,RT^2$
13/9 : $RT\,RT^4\,RT^2\,RT^2\,RT^2$
14/9 : $RT\,RT^3\,RT^5$
16/9 : $RT\,RT^3\,RT^2\,RT^2\,RT^3$
17/9 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
1/10 : RT^{10}
3/10 : $RT^4\,RT^2\,RT^2$
7/10 : $RT^2\,RT^2\,RT^4$
9/10 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
11/10 : $RT\,RT^{11}$
13/10 : $RT\,RT^5\,RT^2\,RT^2$
17/10 : $RT\,RT^3\,RT^2\,RT^4$
19/10 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
21/10 : $RT\,RT^2\,RT^{11}$
23/10 : $RT\,RT^2\,RT^5\,RT^2\,RT^2$
27/10 : $RT\,RT^2\,RT^3\,RT^2\,RT^4$
29/10 : $RT\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
1/11 : RT^{11}
2/11 : $RT^6\,RT^2$
3/11 : $RT^4\,RT^3$
4/11 : $RT^3\,RT^4$
5/11 : $RT^3\,RT^2\,RT^2\,RT^2\,RT^2$
6/11 : $RT^2\,RT^6$
7/11 : $RT^2\,RT^3\,RT^2\,RT^2$
8/11 : $RT^2\,RT^2\,RT^3\,RT^2$
9/11 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^3$
10/11 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
12/11 : $RT\,RT^{12}$
13/11 : $RT\,RT^7\,RT^2$
1/12 : RT^{12}
5/12 : $RT^3\,RT^2\,RT^3$
7/12 : $RT^2\,RT^4\,RT^2$
11/12 : $RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
13/12 : $RT\,RT^{13}$
17/12 : $RT\,RT^4\,RT^2\,RT^3$
19/12 : $RT\,RT^3\,RT^4\,RT^2$
23/12 : $RT\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$
25/12 : $RT\,RT^2\,RT^{13}$
29/12 : $RT\,RT^2\,RT^4\,RT^2\,RT^3$
31/12 : $RT\,RT^2\,RT^3\,RT^4\,RT^2$
35/12 : $RT\,RT^2\,RT^3\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2\,RT^2$

$-1/1:$	T
$-2/1:$	T^2
$-3/1:$	T^3
$-4/1:$	T^4
$-5/1:$	T^5
$-6/1:$	T^6
$-7/1:$	T^7
$-8/1:$	T^8
$-9/1:$	T^9
$-10/1:$	T^{10}
$-11/1:$	T^{11}
$-12/1:$	T^{12}
$-1/2:$	TRT^2
$-3/2:$	T^2RT^2
$-5/2:$	T^3RT^2
$-7/2:$	T^4RT^2
$-9/2:$	T^5RT^2
$-11/2:$	T^6RT^2
$-13/2:$	T^7RT^2
$-15/2:$	T^8RT^2
$-17/2:$	T^9RT^2
$-19/2:$	$T^{10}RT^2$
$-21/2:$	$T^{11}RT^2$
$-23/2:$	$T^{12}RT^2$
$-1/3:$	TRT^2RT^2
$-2/3:$	TRT^3
$-4/3:$	$T^2RT^2RT^2$
$-5/3:$	T^2RT^3
$-7/3:$	$T^3RT^2RT^2$
$-8/3:$	T^3RT^3
$-10/3:$	$T^4RT^2RT^2$
$-11/3:$	T^4RT^3
$-13/3:$	$T^5RT^2RT^2$
$-14/3:$	T^5RT^3
$-16/3:$	$T^6RT^2RT^2$
$-17/3:$	T^6RT^3
$-1/4:$	$TRT^2RT^2RT^2$
$-3/4:$	TRT^4
$-5/4:$	$T^2RT^2RT^2RT^2$
$-7/4:$	T^2RT^4
$-9/4:$	$T^3RT^2RT^2RT^2$
$-11/4:$	T^3RT^4
$-13/4:$	$T^4RT^2RT^2RT^2$
$-15/4:$	T^4RT^4
$-17/4:$	$T^5RT^2RT^2RT^2$
$-19/4:$	T^5RT^4
$-21/4:$	$T^6RT^2RT^2RT^2$
$-23/4:$	T^6RT^4
$-1/5:$	$TRT^2RT^2RT^2RT^2$
$-2/5:$	TRT^2RT^3
$-3/5:$	TRT^3RT^2
$-4/5:$	TRT^5
$-6/5:$	$T^2RT^2RT^2RT^2RT^2$
$-7/5:$	$T^2RT^2RT^3$
$-8/5:$	$T^2RT^3RT^2$
$-9/5:$	T^2RT^5
$-11/5:$	$T^3RT^2RT^2RT^2RT^2$
$-12/5:$	$T^3RT^2RT^3$
$-13/5:$	$T^3RT^3RT^2$
$-14/5:$	T^3RT^5
$-1/6:$	$TRT^2RT^2RT^2RT^2RT^2$
$-5/6:$	TRT^6
$-7/6:$	$T^2RT^2RT^2RT^2RT^2RT^2$
$-11/6:$	T^2RT^6
$-13/6:$	$T^3RT^2RT^2RT^2RT^2RT^2$
$-17/6:$	T^3RT^6
$-19/6:$	$T^4RT^2RT^2RT^2RT^2RT^2$
$-23/6:$	T^4RT^6
$-25/6:$	$T^5RT^2RT^2RT^2RT^2RT^2$
$-29/6:$	T^5RT^6
$-31/6:$	$T^6RT^2RT^2RT^2RT^2RT^2$
$-35/6:$	T^6RT^6

$-1/7:$	$TRT^2RT^2RT^2RT^2RT^2RT^2$
$-2/7:$	$TRT^2RT^2RT^3$
$-3/7:$	TRT^2RT^4
$-4/7:$	$TRT^3RT^2RT^2$
$-5/7:$	TRT^4RT^2
$-6/7:$	TRT^7
$-8/7:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-9/7:$	$T^2RT^2RT^2RT^3$
$-10/7:$	$T^2RT^2RT^4$
$-11/7:$	$T^2RT^3RT^2RT^2$
$-12/7:$	$T^2RT^4RT^2$
$-13/7:$	T^2RT^7
$-1/8:$	$TRT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-3/8:$	$TRT^2RT^3RT^2$
$-5/8:$	TRT^3RT^3
$-7/8:$	TRT^8
$-9/8:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-11/8:$	$T^2RT^2RT^3RT^2$
$-13/8:$	$T^2RT^3RT^3$
$-15/8:$	T^2RT^8
$-17/8:$	$T^3RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-19/8:$	$T^3RT^2RT^3RT^2$
$-21/8:$	$T^3RT^3RT^3$
$-23/8:$	T^3RT^8
$-1/9:$	$TRT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-2/9:$	$TRT^2RT^2RT^2RT^3$
$-4/9:$	TRT^2RT^5
$-5/9:$	$TRT^3RT^2RT^2RT^2$
$-7/9:$	TRT^5RT^2
$-8/9:$	TRT^9
$-10/9:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-11/9:$	$T^2RT^2RT^2RT^2RT^3$
$-13/9:$	$T^2RT^2RT^5$
$-14/9:$	$T^2RT^3RT^2RT^2RT^2$
$-16/9:$	$T^2RT^5RT^2$
$-17/9:$	T^2RT^9
$-1/10:$	$TRT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-3/10:$	$TRT^2RT^2RT^4$
$-7/10:$	$TRT^4RT^2RT^2$
$-9/10:$	TRT^{10}
$-11/10:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-13/10:$	$T^2RT^2RT^2RT^4$
$-17/10:$	$T^2RT^4RT^2RT^2$
$-19/10:$	T^2RT^{10}
$-21/10:$	$T^3RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-23/10:$	$T^3RT^2RT^2RT^4$
$-27/10:$	$T^3RT^4RT^2RT^2$
$-29/10:$	T^3RT^{10}
$-1/11:$	$TRT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-2/11:$	$TRT^2RT^2RT^2RT^2RT^3$
$-3/11:$	$TRT^2RT^2RT^3RT^2$
$-4/11:$	$TRT^2RT^3RT^2RT^2$
$-5/11:$	TRT^2RT^6
$-6/11:$	$TRT^3RT^2RT^2RT^2RT^2$
$-7/11:$	TRT^3RT^4
$-8/11:$	TRT^4RT^3
$-9/11:$	TRT^6RT^2
$-10/11:$	TRT^{11}
$-12/11:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-13/11:$	$T^2RT^2RT^2RT^2RT^2RT^3$
$-1/12:$	$TRT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-5/12:$	$TRT^2RT^4RT^2$
$-7/12:$	$TRT^3RT^2RT^3$
$-11/12:$	TRT^{12}
$-13/12:$	$T^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-17/12:$	$T^2RT^2RT^4RT^2$
$-19/12:$	$T^2RT^3RT^2RT^3$
$-23/12:$	T^2RT^{12}
$-25/12:$	$T^3RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2RT^2$
$-29/12:$	$T^3RT^2RT^4RT^2$
$-31/12:$	$T^3RT^3RT^2RT^3$
$-35/12:$	T^3RT^{12}

9 Avoiding Infinity

For some reason, many people are disturbed by the fact that a new number which we called "infinity" has to be added to the rational numbers if we wish to represent all possible tangles. You can use this as a discussion point to remind the students that the process of adding items to our number systems is old and commonly done.

For example, the natural numbers are usually the first system we have, but if you want an inverse for addition that always works, you've got to add the negative numbers to them to create the integers. Then, if you want to be able to invert multiplication (except by zero), you need to add all the rational numbers to the integers. To solve equations like $x^2 = 2$, you find a need to add the algebraic numbers. This continues to the reals and the complex numbers. We're just extending the rationals in a slightly different way to make a number system to represent tangles.

But another way to look at it might be justified by the idea that the numerical operation corresponding to **Rotate** maps x to $-1/x$. If we talk about the slope m of a line in the plane, the slope of a line perpendicular to it has slope $-1/m$: exactly the same operation.

Instead of numbers to represent tangles, use lines from the origin that pass through integer lattice points. This includes, of course, the vetical line (passing through $(0,0)$ and $(0,1)$ that has "undefined" slope, but from a geometric viewpoint, this is just another line).

The **Rotate** command corresponds to rotating the line by $90°$ about the origin.

The **Twist** command is a little bit trickier: to "**Twist**" a line, select a lattice point on the line that has a non-negative x-coordinate (other than $(0,0)$), move that point up by the x-coordinate of the point, and the new line passes through that new point and the origin. Mathematically, if the lattice point has coordinates (x, y), then the new lattice point will have coordinates $(x, y + x)$. This is exactly what we did before: the fraction y/x was converted by a **Twist** command to $y/x + 1 = (y + x)/x$. Check that the right thing happens for the vertical line.

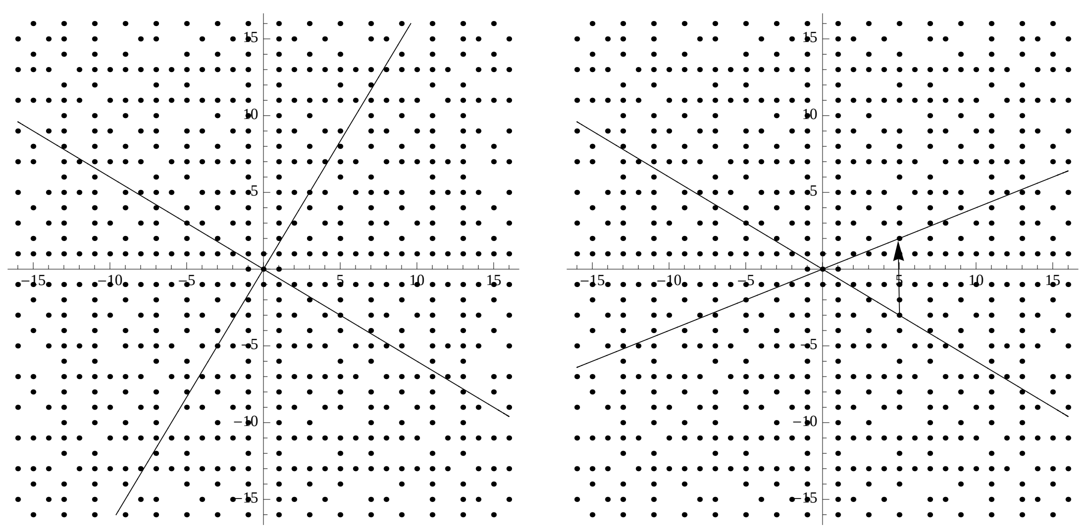

Figure 5: Lines corresponding to tangles

Figure 5 illustrates both operations. On the left side of the figure is a plot of the line having slope $5/3$ and that line rotated $90°$ having slope $-3/5$. If we were trying to reduce to zero the tangle corresponding to $5/3$ we would first issue the **Rotate** command yielding a tangle corresponding to $-3/5$. Next, we would issue a **Twist** command and graphically that corresponds to the operation on the right. A lattice point with $x \geq 0$ on the line is selected (in this case, $(5, -3)$) and that point's x-coordinate is added to its y-coordinate, yielding $(5, 2)$. The new line passes through the origin and $(5, 2)$.

There are a couple of other interesting features that can be seen in the figure. The rational numbers that can be associated with tangles, when reduced to lowest terms will be represented by points that are "visible" from the origin. For example, if you were standing at the origin looking at the point $(3, 5)$, the point $(6, 10)$ is "hidden" behind it. In the figure, only visible points are included.

Second, we apply a **Twist** command only when the line corresponding to our tangle has a negative slope. Since we only increase by the amount corresponding to the x-coordinate, the resulting line, once it has a positive slope, will never have a slope of more than $45°$. When such a line is rotated, the rotation effectively swaps the x and y coordinates (and flips one of the signs), so the resulting corresponding fraction has a smaller denominator.

10 Algebraic Considerations

If we ignore the ropes and just look at the algebra involved, we are basically considering the interaction of two mathematical functions under function composition:

$$\begin{aligned} t(x) &= x + 1 \\ r(x) &= -1/x \end{aligned}$$

If we apply t three times followed by r to any input number, the result is:

$$r(t(t(t(x)))) = r(x + 3) = -1/(x + 3).$$

Note the apparent reversal of the operations due to the functional notation: t is applied first to x, then another application to t to that, and so on.

We can apply any combination of t and r to an input value, in any order, but some applications are "inefficient" in the sense that if we apply r twice in a row, it's as if we did nothing, since $r(r(x)) = x$. It is often useful in mathematics to have a symbol for the "do nothing" operation, or, as it is ususally called, the identity operation. Here we will call that do-nothing operation i: in other words,

$$i(x) = x.$$

We can indicate that fact that the application of r twice in a row is the identity function as:

$$r(r(x)) = i(x).$$

Are there any other simplifications to be found?

The answer is yes, and it is easy to check algebraically that:

$$t(r(t(r(t(r(x)))))) = x = i(x).$$

Here's the proof:

$$
\begin{aligned}
t(r(t(r(t(r(x)))))) &= t(r(t(r(t(-1/x))))) \\
&= t(r(t(r(1-1/x)))) \\
&= t(r(t(x/(1-x)))) \\
&= t(r(1/(1-x))) \\
&= t(x-1) \\
&= x.
\end{aligned}
$$

It is also easy to show that $r(t(r(t(r(t(x)))))) = i(x)$ (note the reversed order). This can be shown with a calculation similar to the one above or by appealing to the associativity of function composition, or by knowing a bit of group theory. The bottom line, however, is that the sequence $RTRTR$ will unto a T. A mathematician would write this as: $T^{-1} = RTRTR$.

This provides a trivial (but often very inefficient) method to undo any sequence of twists and rotates. Imagine that $a, b, c, \dots, y, z$ are either twists or rotates (in any order), and that you have applied the sequence: $abcd \cdots z$ to a tangle. To undo that sequence, you first would undo the last thing you did, namely, z, then you'd undo y, and so on, and finally, undo c, then b, then a. For example, to undo the sequence $T^3RT = TTTRT$, you would apply:

$$T^{-1}R^{-1}T^{-1}T^{-1}T^{-1} = (RTRTR)R(RTRTR)(RTRTR)(RTRTR).$$

Notice, of course, that this sequence contains places where more than one R is applied at a time, and since each such R can undo the previous one, we obtain:

$$
\begin{aligned}
(RTRTR)R(RTRTR)(RTRTR)(RTRTR) &= RTRTR^3TRTR^2TRTR^2TRTR \\
&= RTRTRTRT^2RT^2RTR.
\end{aligned}
$$

But the term on the right can be simplified even more, since the $RTRTRT$ on the left does nothing. Thus, the inverse of T^3RT is RT^2RT^2RTR.

One nice way to illustrate this physically is using a solid piece of wood with four cords attached to the corners as in Figure 6.

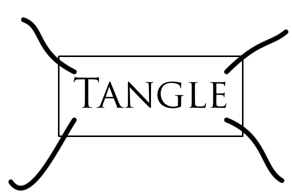

Figure 6: Tangle Board

The "tangle" is a chunk of wood, so it can never be untangled, and you can hand one end to each of four persons as was done in earlier, beginning with the text "TANGLE" facing the audience and right-side up. Then do a $TRTRTR$ and see that the board returns exactly to its original orientation.

10.1 Group Theory

All of the above can be used as an introduction to a subject called "Group Theory". Volumes are written about the subject, but the operations T and R and their combinations form an infinite group. A group is a mathematical object that consists of a set of objects G (in this case, the various combinations of R and T), and an operation $*$ on those objects (in this case, their combination), satisfying the following four axioms:

1. The operation $*$ is closed. In other words, if A and B are any two objects in G, then $A * B$ is also in G.
2. The operation $*$ is associative. In other words, if A, B and C are any three objects in G, then $(A * B) * C = A * (B * C)$.
3. There exists an identity object I in G. In other words, if A is any object in G, then $A * I = I * A = A$.
4. For every object A in G, there is an inverse A^{-1} in G such that $A * A^{-1} = A^{-1}A = I$.

In the case of our tangle group, the elements in G are simply finite lists of R and T. Inverses can be calculated as described above, and the identity is the "do nothing" operation, or equivalently, RR. We have omitted the operation $*$ in our description, but you can imagine it being between any pair of letters, so $RTTRT$ could have been written $R * T * T * R * T$.

What makes our group a bit more interesting is that arbitrary strings of R and T can often be simplified because of the conditions $RR = I$ and $(TR)^3 = I$.

In the last section, we said that "the inverse of T^3RT is RT^2RT^2RTR". Let's see why.

If it is the inverse, then applying one followed by the other should yield the identity. In other words, it should be true that:

$$T^3RTRT^2RT^2RTR = I,$$

and we should be able to show that it is using only the two identities $R^2 = I$ and $(TR)^3 = I$ (which is equivalent to (why?) $(RT)^3 = I$). These last two can also be written: $T = RTRTR$ and $R = TRTRT$, so:

$$\begin{aligned}
T^3RTRT^2RT^2RTR &= TTTRTRTTRTTRTR \\
&= TT(TRTRT)TRTTRTR \\
&= TT(R)TRTTRTR \\
&= T(TRTRT)TRTR \\
&= T(R)TRTR = (TR)^3 = I
\end{aligned}$$

If you happen to already know something about group theory, then the tangle group is technically the free group on two generators, R and T, modulo the following identities: $R^2 = I$ and $(TR)^3 = I$.

11 Sample Tangles

When you run this circle for the first few times, it's easy to make arithmetic mistakes, since you need to do arithmetic while you're trying to do everything else. What follows are a few examples that you can do by rote: have the kids do the given sequence, and make sure that the solution sequence is the one described afterwards.

The "Sequence" begins with $0 = 0/1$ and shows how each step generates the next fraction. The fraction at the end represents the result of the entire sequence. The "Solution" represents the shortest correct sequence that will return the ropes to the solved (0) state.

Sequence: TTT

$$\frac{0}{1} \xrightarrow{T} \frac{1}{1} \xrightarrow{T} \frac{2}{1} \xrightarrow{T} \frac{3}{1}$$

Solution: RTRTTRTT

$$\frac{3}{1} \xrightarrow{R} \frac{-1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} \frac{-3}{2} \xrightarrow{T} \frac{-1}{2} \xrightarrow{T} \frac{1}{2} \xrightarrow{R} \frac{-2}{1} \xrightarrow{T} \frac{-1}{1} \xrightarrow{T} \frac{0}{1}$$

Sequence: TTTRT

$$\frac{0}{1} \xrightarrow{T} \frac{1}{1} \xrightarrow{T} \frac{2}{1} \xrightarrow{T} \frac{3}{1} \xrightarrow{R} \frac{-1}{3} \xrightarrow{T} \frac{2}{3}$$

Solution: RTTRTT

$$\frac{2}{3} \xrightarrow{R} \frac{-3}{2} \xrightarrow{T} \frac{-1}{2} \xrightarrow{T} \frac{1}{2} \xrightarrow{R} \frac{-2}{1} \xrightarrow{T} \frac{-1}{1} \xrightarrow{T} \frac{0}{1}$$

Sequence: TTTRTTTTR

$$\frac{0}{1} \xrightarrow{T} \frac{1}{1} \xrightarrow{T} \frac{2}{1} \xrightarrow{T} \frac{3}{1} \xrightarrow{R} \frac{-1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{T} \frac{5}{3} \xrightarrow{T} \frac{8}{3} \xrightarrow{T} \frac{11}{3} \xrightarrow{R} \frac{-3}{11}$$

Solution: TRTTRTTRTTTRTT

$$\frac{-3}{11} \xrightarrow{T} \frac{8}{11} \xrightarrow{R} \frac{-11}{8} \xrightarrow{T} \frac{-3}{8} \xrightarrow{T} \frac{5}{8} \xrightarrow{R} \frac{-8}{5} \xrightarrow{T} \frac{-3}{5} \xrightarrow{T} \frac{2}{5} \xrightarrow{R} \frac{-5}{2} \xrightarrow{T} \frac{-3}{2} \xrightarrow{T} \frac{-1}{2}$$

$$\xrightarrow{T} \frac{1}{2} \xrightarrow{R} \frac{-2}{1} \xrightarrow{T} \frac{-1}{1} \xrightarrow{T} \frac{0}{1}$$

Sequence: TTTTRTTTT

$$\frac{0}{1} \xrightarrow{T} \frac{1}{1} \xrightarrow{T} \frac{2}{1} \xrightarrow{T} \frac{3}{1} \xrightarrow{T} \frac{4}{1} \xrightarrow{R} \frac{-1}{4} \xrightarrow{T} \frac{3}{4} \xrightarrow{T} \frac{7}{4} \xrightarrow{T} \frac{11}{4} \xrightarrow{T} \frac{15}{4}$$

Solution: RTRTTRTTRTTTRTTRTT

$$\frac{15}{4} \xrightarrow{R} \frac{-4}{15} \xrightarrow{T} \frac{11}{15} \xrightarrow{R} \frac{-15}{11} \xrightarrow{T} \frac{-4}{11} \xrightarrow{T} \frac{7}{11} \xrightarrow{R} \frac{-11}{7} \xrightarrow{T} \frac{-4}{7} \xrightarrow{T} \frac{3}{7} \xrightarrow{R} \frac{-7}{3} \xrightarrow{T} \frac{-4}{3}$$

$$\xrightarrow{T} \frac{-1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} \frac{-3}{2} \xrightarrow{T} \frac{-1}{2} \xrightarrow{T} \frac{1}{2} \xrightarrow{R} \frac{-2}{1} \xrightarrow{T} \frac{-1}{1} \xrightarrow{T} \frac{0}{1}$$

If you are confident of your arithmetic, I have found that a nice tangle to use is $-17/43$. Using the methods of Section 8.4, start with $17/43$ on the blackboard and grind that down to zero (using only math on the blackboard). Then reverse the steps, starting from a zero tangle to obtain a tangle represented by $-17/43$. It is a fairly long process, but it builds the suspense. Tie a bag over this tangle, and the nice thing is that it untangles quickly, and in fact, ends with nine **Twist** commands in a row!