J. Aebly, Démonstration du problème du scrutin par des considérations géométriques, *L'enseignement mathématique* 23 (1923) 185–186.

Proof of the ballot problem by geometric considerations
by
J. Aebly (Zurich)

The problem is as follows: Two candidates, A and B, are present; a well informed voter knows in advance that A will receive m votes, and B will receive n votes, with $m > n$. One requires the probability that A will keep the majority throughout the counting of the ballots.

Poincaré gives, in his Theory of Probability, an elegant solution of the problem due to D. André, which however is rather long (p. 45 - 49). Having arrived by geometric considerations at a shorter (though no less rigorous) solution to the problem, I will describe it in what follows.

We make, corresponding to the total possible ways of the counting the ballots, a rectangle whose sides have lengths $(m + 1)$ and $(n + 1)$ units respectively, that we will divide into $(m + 1)(n + 1)$ squares by lines parallel to the sides of the rectangle. Each square will be represented by a double index (i, k) whose first term indicates the row, and the second the column to which the square belongs, the range is from $(0, 0)$ to (n, m).

Let us imagine that one must travel from $(0, 0)$ with (n, m) by an unspecified way satisfying the following condition: the passage from one square to another can be made in only two manners, either to the square located below, or to the square located on the right, i.e. the only allowed possibilities are to change (i, k) to $(i + 1, k)$ or to $(i, k + 1)$. The square (i, k) could then be considered as representing all the ways of moving from $(0, 0)$ to (i, k), i.e. the set of all ways of counting i ballots for B and k ballots for A.

These conventions established, we proceed to the solution of the problem. The total number of outcomes represented by (n, m) equals $\binom{m+n}{n}$. The successful outcomes are represented by all the paths not touching any squares whose two indices have the same number, such as (i, i). Since (i, i) can be approached by only two squares because of the established principle, and since there is complete symmetry with respect to the diagonal passing from $(0, 0)$ to (n, n) by the squares such as (i, i), it follows that each path coming from one side corresponds to one and only one path coming from other side.

One of these groups [i.e., sets of paths] can be regarded as starting from $(1,0)$, the other from $(0,1)$.

The first represents all the cases where A loses the majority with the first vote and the other the cases where A initially has the majority but loses it at (i,i). B necessarily having to lose the majority, the paths starting from $(1,0)$ must necessarily pass through one of the squares (i,i). It is enough to consider the first passage [i.e., the first time the path touches the diagonal] for the paths emanating from $(0,1)$. The two groups thus comprise all the unfavorable cases. Their number being $2\binom{m+n-1}{m}$, the required probability is thus

$$1 - 2\frac{\binom{m+n-1}{m}}{\binom{m+n}{m}} = 1 - \frac{2n}{m+n}, \text{ or } \frac{m-n}{m+n}.$$