D. André, Solution directe du problème résolu par M. Bertrand, *Comptes Rendus de l'Académie des Sciences*, Paris 105 (1887) 436–437.

THEORY OF PROBABILITY. — *Direct solution of the problem solved by Mr.* Bertrand. Note of Mr. **Désiré Andre**.

Suppose that two candidates A and B are in an election. The number of the voters is $\alpha + \beta$. A receives α votes and is elected, and B receives β votes. One desires the probability that, during the counting of the votes, the number of votes for A always exceeds that for B.

The number of possible outcomes is obviously the number of permutations one can form with a letters A and β letters B.

Let $Q_{\alpha,\beta}$ be the number of the *unfavorable* outcomes. The permutations corresponding to them are of two kinds: those that start with B, and those that start with A.

The number of unfavorable permutations starting with B equals the number of all permutations which one can form with α letters A and $\beta-1$ letters B, because it is obviously enough to suppress the initial letter B to obtain the remaining letters.

The number of unfavorable permutations starting with A is the same as above, because one can, by a simple rule make a one-to-one correspondence with the permutations formed with α letters A and $\beta - 1$ letters B.

This rule is composed of two parts:

1) Given an unfavorable permutation starting with A, one removes the first occurrence of B that violates the law of the problem [causes the number of B's to equal the number of A's], then one exchanges the two groups separated by this letter: one obtains thus a permutation, uniquely determined, of α letters A and $\beta - 1$ letters B. Consider, for example, the unfavorable permutation $AABBABAA$, of five letters A and three letters B; by removing the first B that violates the law, one separates two groups AAB, $ABAA$; by exchanging these groups, one obtains the permutation $ABAAAAB$, formed of five letters A and two letters B.

2) Given an arbitrary permutation of α letters A and $\beta-1$ letters B, one traverses it from right to left until one obtains a group where the number of A's exceeds [by one] the number of B's; one considers this group and that which the letters placed at its left form; one exchanges these two groups,

while placing between them a letter B: one thus forms an unfavorable permutation starting with A and uniquely given. Consider, for example, the permutation $ABAAAAB$; while operating as described, one divides it in two groups $ABAA$, AAB; by exchanging these groups and placing the letter B between them, one forms the unfavorable permutation $AABBABAA$.

It results from all the above that the total number of unfavorable outcomes is twice the number of permutations one can form with α letters A and $\beta - 1$ letters B; i.e.,

$$Q_{\alpha,\beta} = 2\frac{(\alpha+\beta-1)!}{\alpha!(\beta-1)!}$$

If one indicates by $Q_{\alpha,\beta}$ the number of the favorable outcomes, one thus has

$$P_{\alpha,\beta} = \frac{(\alpha+\beta)!}{\alpha!\beta!} - 2\frac{(\alpha+\beta-1)!}{\alpha!(\beta-1)!}$$

or

$$P_{\alpha,\beta} = \frac{(\alpha+\beta-1)!}{\alpha!\beta!}(\alpha-\beta).$$

Consequently, the required probability is

$$\frac{\alpha-\beta}{\alpha+\beta}.$$