Lost in Translation: A Reflection on the Ballot Problem and André's Original Method

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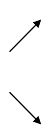
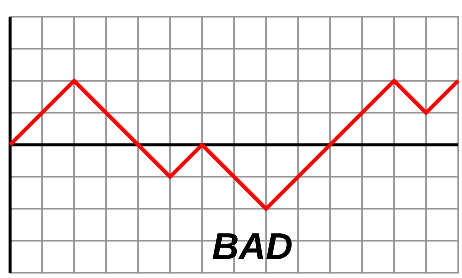
The Ballot Problem (1887)

In how many ways can ***a*** **upsteps** and ***b*** **downsteps** be ordered so that no step ends on or below the *x*-axis?

$a = 8$

$b = 6$





Joseph Bertrand
1822 - 1900

The number of good paths is

$$\frac{a-b}{a+b}\binom{a+b}{a}$$

Bertrand asked "Is there a *direct* proof?"

Désiré André (1887)

Solves the ballot problem!

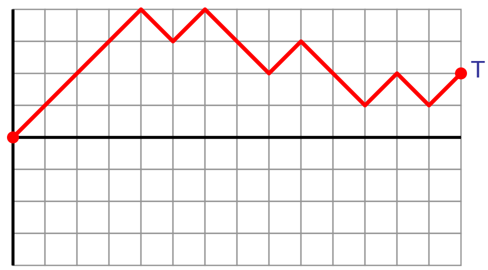
And mathematicians celebrate!

Désiré André

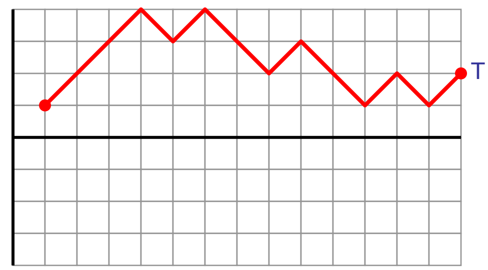
1840 - 1917

Today, the most famous solution to the ballot problem is **André's Reflection Method**…

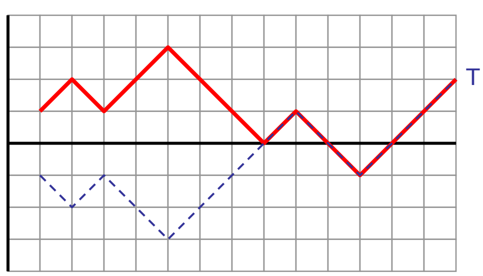
Number of good paths = Number of good paths from (1, 1) to T.

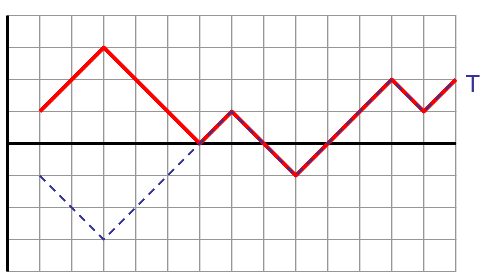


Terminal point T has coordinates (a+b, a-b)



Trick: count the number of *bad* paths from (1,1) to T.





Bad paths from (1,1) to T

↓ ↑

All paths from (1, -1) to T

Number of good paths from (0,0)

= Number of good paths from (1,1)

= [Total number of paths from (1,1)]

- [Number of bad paths from (1,1)]

reflection

= [Total number of paths from (1,1)]

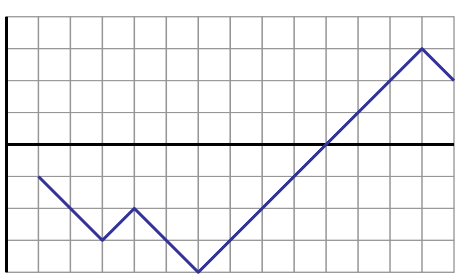
- [Total number of paths from (1,-1)]

Total # of paths from (1,1):

a - 1 upsteps

b downsteps

Total: $\binom{a+b-1}{b}$



Total # of paths from (1,-1):

a upsteps

b - 1 downsteps

Total: $\binom{a+b-1}{a}$

Solution to the Ballot Problem:

$$\binom{a+b-1}{b}-\binom{a+b-1}{a}=\cdots=\frac{a-b}{a+b}\binom{a+b}{a}$$

The celebrated reflection method of André…

MathWorld

I.P. Goulden and Luis G. Serrano, Maintaining the Spirit of the Reflection Principle when the Boundary has Arbitrary Integer Slope, J. Combinatorial Theory (A) 104 (2003) 317-326. "André gave a direct geometric bijection between the subset of bad paths and the set A of all paths from (1, -1) to (m, n), and the result then follows immediately…"

J.H. Van Lint and R.M. Wilson, A Course in Combinatorics, Cambridge University Press, 2001. p. 151: "The reflection principle of Fig. 14.2 was used by the French combinatorialist D. André (1840-1917) in his solution of Bertrand's famous ballot problem…"

I. Karatzas and S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, 1998. They write "Here is the argument of Désiré André…" and proceed with the reflection method.

H. Bauer, Probability Theory, Walter de Gruyter, Berlin, New York, 1996. p. 231: "In the literature, this reflection principle is usually attributed to D. André (1840-1918). It occurs in the form of such a geometric argument in André [1887]."

P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, Math. Intelligencer. 13 (1991) 64–75.

D. Stanton and D. White, Constructive Combinatorics, Springer-Verlag, New York, 1986.

D. Zeilberger, André's reflection proof generalized to the many-candidate ballot problem, Discrete Mathematics 44 (1983) 325-326.

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The problem is…

A Recent Discovery

André never used
the reflection method!

What André did:

1. Count # bad ballot permutations.
2. Subtract that from the total # of permutations to get # of good permutations.

How André counted bad outcomes…

André’s Actual Method

Ballots are marked with “A” or “B”.

Two categories of bad ballot permutations:

- Those that start with A ← Next slide…
- Those that start with B

Easy: *every* permutation starting with B is bad. There are $\binom{a+(b-1)}{a}$ of these.

Claim:

# of bad permutations starting with A

= # of all permutations with a A’s and (b − 1) B’s.

A A B B A B A A

A A B   A B A A

A B A A   A A B

A B A A A A B

Given a bad permutation starting with A…

Find the first bad B

Remove it

Exchange the two parts

Done!

Now reverse the process…

Claim:

# of bad permutations starting with A = # of all permutations with a A's and (b – 1) B's.

A B A A A A B

A B A A   A A B

A A B   A B A A

Given a permutation with a A's and (b – 1) B's…

Scan from right until A's exceed B's (by 1).

Exchange the two parts

Insert B

Done!

A A B **B** A B A A

Thus $\binom{a+(b-1)}{a}$ bads start with A

Bad permutations:

- Those that start with A
- Those that start with B

$$2\binom{a+(b-1)}{a}$$

Good ballot permutations:

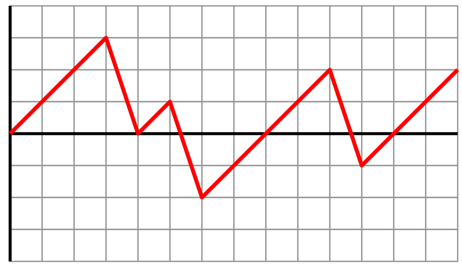
$$\binom{a+b}{a} - 2\binom{a+(b-1)}{a} = \frac{a-b}{a+b}\binom{a+b}{a}$$

- No geometry
- No reflection (transposing A's and B's)

The Generalized Ballot Problem

Fix a positive integer k.

How many paths with a 1-unit upsteps and b k-unit downsteps have no step ending on or below the x-axis?

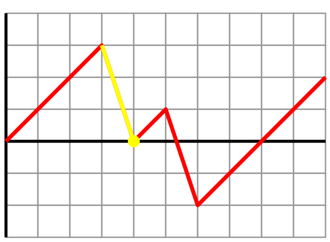


k = 3

$$\frac{a-kb}{a+b}\binom{a+b}{a}$$

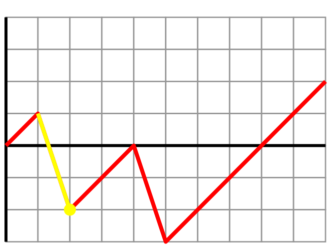
- The reflection method does not generalize.
- André’s original method does!

k = 3. Classify bad paths: $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$.

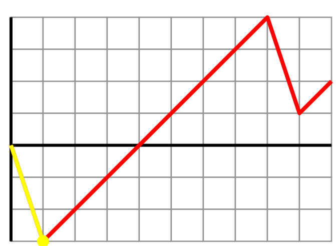


A path in $\mathcal{B}_0$

A path in $\mathcal{B}_1$



A path in $\mathcal{B}_2$



A path in $\mathcal{B}_3$

For arbitrary k we create $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_k$.

Fact: These sets all have the same size!

By André's find-bad-step-remove-it-exchange-two-sides trick, each set has size

$$\binom{a+(b-1)}{a}$$

Thus, the number of bad paths is $(k+1)\binom{a+(b-1)}{a}$

Thus, the number of good paths is

$$\binom{a+b}{a} - (k+1)\binom{a+(b-1)}{a} = \frac{a-kb}{a+b}\binom{a+b}{a}$$

Concluding Thoughts

- So where and when did the reflection method originate?
  - Aebly 1923? 1915?
- When did André start getting credit for the reflection method?
  - 1950's ? Earlier?

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Lost (and Found) in Translation: André's Actual Method and its Application to the Generalized Ballot Problem