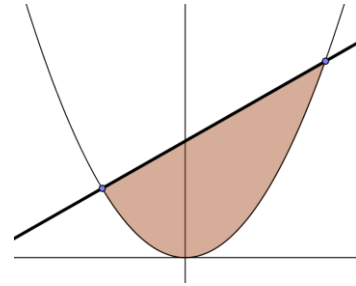


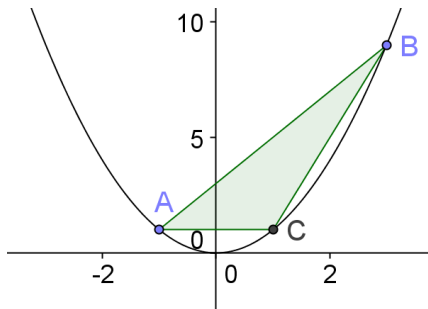
Archimedes discovers the area of a parabolic segment.

Archimedes (287-212 BCE) is one of the most revered mathematicians of all time. Math historian William Dunham writes, “Archimedes stands unchallenged as the greatest mathematician of antiquity. His results, which survive in a dozen books and fragments, are of the highest quality and show a logical sophistication and polish that is truly astounding.” In his treatise *Quadrature of the Parabola* he finds the area of a parabolic segment.

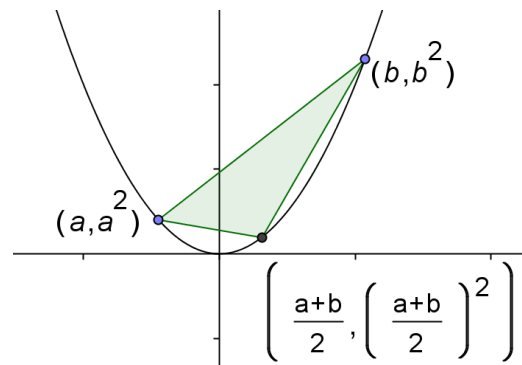
1. Goal: Find the area of a *parabolic segment*, that is, the area enclosed by a parabola and a straight line. We'll find the answer using calculus, then we'll follow the method of Archimedes.



2. Below is a graph of the parabola $y = x^2$ along with the points $A = (-1, 1)$ and $B = (3, 9)$. Find the point C on the graph such that the x -value of C is midway between the x -value of A and the x -value of B .



3. Find the area of triangle ABC .
4. Use calculus to find the area of the region bounded by the parabola and the line segment AB .
5. Fill in the blank: the area of the parabolic segment is ____ times as large as the area of the triangle.
6. We'll consider a more general setting, now. Suppose a line intersects the parabola $y = x^2$ at the points $A = (a, a^2)$ and $B = (b, b^2)$ (assume $a < b$). We pick a third point, as before, at $C = \left(\frac{a+b}{2}, \left(\frac{a+b}{2}\right)^2\right)$.



7. Show that the area of triangle ABC is $\frac{1}{8}(b - a)^3$.

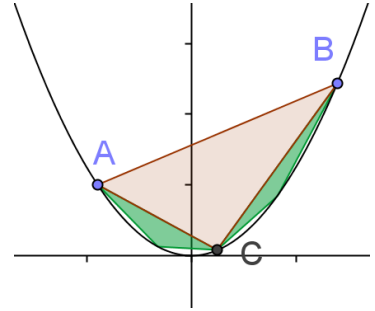
[Hint: make three trapezoids by drawing vertical lines from A , B , and C down to the x -axis. Subtract the areas of the two small trapezoids from the one big trapezoid.]

Does your answer agree with step #3 above?

8. Use calculus to find the area of the parabolic segment. Does your answer agree with your result in step #5 above?

9. Now we start Archimedes' approach to showing that the area of the parabolic segment is $\frac{4}{3}$ the area of the triangle. Notice an amazing fact that follows from #7: The area of the triangle depends only on the *difference* between a and b , and not the actual values of a and b .

Create two smaller triangles to "fill in the gaps" as shown in the picture. If the area of the big triangle is T , then what is the area of each of the two smaller, darker triangles? (Notice that each one is only $\frac{1}{2}$ as wide as the one big triangle).



10. Next, we can "fill gaps" with 4 even smaller triangles, each half as wide again, then 8 very small triangles, then 16 very *very* small triangles, and so on... Justify: If the area of the one initial big triangle is T , then the area of the parabolic segment is

$$T \left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots \right)$$

11. Prove that $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{4}{3}$, and you've completed the Archimedes proof!

Postscript: Archimedes didn't have the advantage of Cartesian coordinates (invented in the early 1600's) and you can imagine how this complicates the process of finding the area of the triangle in the parabola. Also, although Euclid had earlier shown how to compute the sum of a geometric series, Archimedes didn't follow Euclid's method, but instead created his own way for summing a geometric series. Finally, this trick of filling in a regions with infinitely many smaller shapes, each getting smaller and smaller is called the "method of exhaustion" and is attributed to Eudoxus (408-355 BCE). Archimedes, however, took the method to new heights, masterfully using it in many of his greatest proofs.

Cardano and Bombelli and the solution of cubic equations

The problem of solving equations has been around since antiquity. What value of x makes the equation $5x = 2$ true? Linear equations are easy, but quadratic equations are a little harder: what value(s) of x satisfies the equation $5x^2 + 3x = 2$? The quadratic equation (at least a rudimentary form of it) goes back to the Babylonians c. 2000 BCE: the values of x that satisfy the equation $ax^2 + bx + c = 0$, are $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$. Amazingly, the solutions of cubic equations would not be found until the Renaissance. Sixteenth century Italy was the setting for some very entertaining stories from the history of algebra. In 1545, Gerolamo Cardano published *Ars Magna*, in which he showed that a solution to the cubic equation $x^3 = cx + d$ is

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}}$$

1. Verify that this does indeed work for the equation $x^3 = 3x + 18$.
2. What strange thing happens with the equation $x^3 = 15x + 4$?

Cardano did not know what to do with this case, but Rafael Bombelli took the challenge. He “invented” imaginary numbers to save the day.

3. Verify the following statement about **complex numbers**: $(2 + i)^3 = 2 + 11i$. (Here you just have to know that $i^2 = -1$.)
4. While you are at it, verify the fact that, $(2 - i)^3 = 2 - 11i$.
5. Show how this allows us to use Cardano’s formula even in this case.

Postscript: Cardano was a crazy person; read about him on the *MacTutor History of Mathematics* website.

In the 1500’s negative numbers were treated with suspicion and avoided whenever possible. Square roots of negative numbers were downright preposterous. It is commonly, but mistakenly, thought that imaginary numbers were introduced as a tool for solving quadratics. In fact, they were first used as a tool for solving cubics, as shown here.

Although it was stated differently in *Ars Magna*, Cardano provided a solution to the “general” cubic equation $ax^3 + bx^2 + cx + d = 0$. In that same book, his pupil Ludovico Ferrari showed how to solve the general quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$. At this point the hunt was on among mathematicians to find solutions for the general quintic equation. A shock came in the mid-1820’s when Norwegian Niels Abel proved that *there is no algebraic solution to the general quintic equation*; in other words, there is no way we can manipulate the coefficients a, b, c, d, e, f in the equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ using the rules of algebra (add, subtract, multiply, divide, take powers, extract roots) to produce a solution to the equation. Amazing!

Newton Generalizes the Binomial Theorem

Isaac Newton (1642-1727) is regarded as the greatest mathematician of the modern era, perhaps rivaled only by Archimedes for the title of greatest mathematician ever. His *Philosophiæ Naturalis Principia Mathematica*, published in 1687, is considered to be one of the most influential books in the history of science. Here we look at one of his very first mathematical discoveries, the generalized binomial theorem.

1. Expand the following powers of $1 + x$:

- a. $(1 + x)^1 =$

- b. $(1 + x)^2 =$

- c. $(1 + x)^3 =$

2. The Binomial Theorem states that

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

where

$$\binom{n}{k} = \frac{n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1)}{k!}$$

for positive integers n and k . Verify this formula for the coefficients you found in step 1. (Notice that the numerator in the above expression has k factors.)

3. Newton wanted to figure out a similar way to expand $(1 + x)^{1/2}$. Here's Newton's idea: if $f(x) = (1 + x)^{1/2}$, then $(f(x))^2 = 1 + x$.
 - a. Find a **linear** function $f(x)$ such that $(f(x))^2 = 1 + x +$ other stuff of degree 2 or higher.
 - b. Find a **quadratic** function $f(x)$ such that $(f(x))^2 = 1 + x +$ other stuff of degree 3 or higher.
 - c. Find a **cubic** function $f(x)$ such that $(f(x))^2 = 1 + x +$ other stuff of degree 4 or higher.
4. Use your answers to #3 above to create a Generalized Binomial Theorem for $n = \frac{1}{2}$. How should $\binom{1/2}{k}$ be defined?
5. Use your formula from #4 to find a **quartic** function $f(x)$ such that $(f(x))^2 = 1 + x +$ other stuff of degree 5 or higher.

6. Use your formula from #4 to write out the first 8 terms of $(1 + x)^{1/2}$.
7. Newton used his theorem to approximate square roots. Use your result to #6 (and a calculator) to approximate $\sqrt{3}$. Compare this with your calculator's approximation for $\sqrt{3}$.

Postscript: Newton discovered the generalized binomial theorem around 1665. Not to depress you, but he was your age then. Don't worry though – Newton makes us all look bad. We'll see that Newton's generalized binomial theorem plays a key role in his attempt to approximate π . By the way, the notation $\binom{n}{k}$ was first introduced in 1826, so Newton's real calculations were a bit messier.

One last idea – Newton's expansion of $(1 + x)^{1/2}$ is the same as the Taylor series expansion, expanded around $x = 0$. It's worth pondering for a while whether these two expansions MUST, *a priori*, be the same.

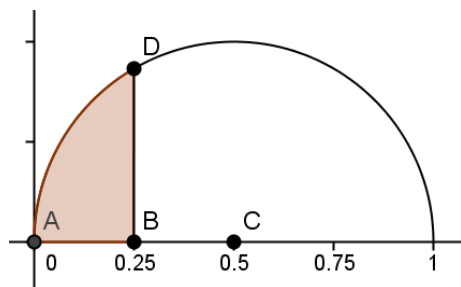
Newton Approximates π

A circle of diameter 1 has a circumference of π . Archimedes approximated π by inscribing and circumscribing polygons about a circle of diameter 1. He found that an inscribed 96-gon has a perimeter greater than $3\frac{10}{71}$ and a circumscribed 96-gon has a perimeter less than $3\frac{1}{7}$, thus, $3\frac{10}{71} < \pi < 3\frac{1}{7}$. For centuries other mathematicians improved on Archimedes's approximation by using polygons with more and more sides.

- In 1593, Adriaen Von Roomen approximated π to 15 decimal places using polygons with 2^{30} sides!
- In 1610, Ludolph Van Ceulen approximated π to 30 decimal places using polygons with 2^{62} sides!

There had to be a better way!

In the 17th century, an entirely new method of approximating π was found: *series approximation*. Here's how Newton approximated π ...



1. Find the equation of the semicircle in the picture.

- Observe that the “full circle” would satisfy the equation $(x - \frac{1}{2})^2 + (y - 0)^2 = (\frac{1}{2})^2$.
- Solve for y and show that $y = \sqrt{x}\sqrt{1-x}$, in other words, $y = x^{1/2} (1-x)^{1/2}$.
- Use Newton's generalized binomial theorem to show

$$y = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \dots$$

2. Find the area of the shaded region with calculus.

- Now that you have an equation for the semicircle, evaluate a definite integral to show that the area is

$$\frac{1}{12} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots$$

3. Find the area of the shaded region with geometry.

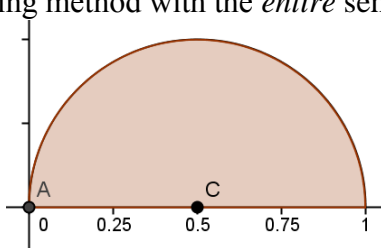
- Find the length of DB .
- Find the area of $\triangle DBC$.
- Notice that $\triangle DBC$ is a 30-60-90 triangle. Use this fact to find the area of sector ACD .
- Notice that (area of shaded region) = (area of sector) – (area of triangle).
Conclude that the area of the shaded region is

$$\frac{\pi}{24} - \frac{\sqrt{3}}{32}$$

4. Set the two expressions for area equal to each other to conclude that

$$\pi = \frac{3\sqrt{3}}{4} + 24 \left[\frac{1}{12} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots \right]$$

5. Use a calculator to approximate π using *five* terms in the series and then using *six* terms in the series. Compare how the accuracy improves.
6. There are a couple good reasons Newton chose the point B to equal $\frac{1}{4}$.
- When calculating area by *calculus*, why is $\frac{1}{4}$ a nice number to use?
 - When calculating area by *geometry*, why is $\frac{1}{4}$ a nice number to use?
7. Let's use Newton's approximating method with the *entire* semicircle shaded.



- Use calculus to show that the semicircle has area $\frac{2}{3} - \frac{1}{5} - \frac{1}{28} - \frac{1}{72} - \dots$.
- Use geometry to show that the semicircle has area $\frac{\pi}{8}$.
- Conclude that

$$\pi = 8 \left[\frac{2}{3} - \frac{1}{5} - \frac{1}{28} - \frac{1}{72} - \dots \right]$$

- The above expression is much simpler than the expression we found in step 4. Why is Newton's original series approximation preferable to this one?

Postscript: Newton correctly approximated π to 15 digits, although he later confessed: "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Mathematicians have often looked at π with curiosity, and sought to find better and better approximations. The motivation is not because better approximation of π are more useful, but rather, the motivation comes from the challenge of creating the best possible mathematical method or algorithm for the approximation. Some highlights:

1706: John Machin was the first to compute 100 decimals of π .

1949: the first computer, ENIAC, computed 2037 digits of π in 70 hours.

1989: 1,011,196,691 digits

2002: 1,241,100,000,000 digits (current record)

The Harmonic Series Diverges (Nicole Oresme, 14th century CE)

Not much happened mathematically in 14th century Europe, but at that time French cleric Nicole Oresme (pronounced *ah-REM*) undertook a detailed study of the mathematics of ratios. His study led him to consider the *harmonic series*: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$.

1. Consider the *geometric series*

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

Euclid (c. 300 BCE) knew how to sum this series. How would *you* add these terms?

2. Consider the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$$

Oresme proved that this “diverges” with the following argument. Explain why each step is true *without actually adding* all the junk on the left side together.

$$1/2 \geq 1/2 \text{ (Brilliant!)}$$

$$1/3 + 1/4 \geq 1/2$$

$$1/5 + 1/6 + 1/7 + 1/8 \geq 1/2$$

$$1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16 \geq 1/2$$

3. Which of the following math-y statements seems to express the gist of Oresme’s argument?

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{m}{2}$$

$$\sum_{n=1}^m \frac{1}{2^n} \geq 1 + \frac{m}{2}$$

$$\sum_{n=1}^{2^m} \frac{1}{n} \geq 1 + \frac{m}{2}$$

4. Using this fact, how many terms of the harmonic series can be added together to get a partial sum that is at least 3? At least 30? At least 300?
5. Consider the infinite series $a_1 + a_2 + a_3 + a_4 + \dots$. Determine whether the following statements are true or false and support your answer.
- If the terms of the series converge to zero, then the series must converge.
 - If the series converges, then the terms must converge to zero.

Postscript: The proof above that the harmonic series diverges is fairly straightforward, easy to follow, and should convince any thoughtful person of its truthness. However, what is so apparent in hindsight is exceedingly difficult to discover on one’s own. (An analogy: a young piano student can certainly play pieces by Mozart, but it takes a genius on the level of Mozart to compose such pieces.) Oresme’s proof that the harmonic series diverges was “lost” (or certainly not well-known) and not until 1689 did another—longer, more complicated—proof appear, penned by the extremely talented brothers Jakob and Johann Bernoulli.