

Biometrika Trust

The Distribution of the Ratio, in a Single Normal Sample, of Range to Standard Deviation

Author(s): H. A. David, H. O. Hartley and E. S. Pearson

Source: *Biometrika*, Vol. 41, No. 3/4 (Dec., 1954), pp. 482-493

Published by: [Biometrika Trust](#)

Stable URL: <http://www.jstor.org/stable/2332728>

Accessed: 07/03/2014 22:30

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Biometrika Trust is collaborating with JSTOR to digitize, preserve and extend access to *Biometrika*.

<http://www.jstor.org>

THE DISTRIBUTION OF THE RATIO, IN A SINGLE NORMAL SAMPLE, OF RANGE TO STANDARD DEVIATION

BY H. A. DAVID, H. O. HARTLEY AND E. S. PEARSON

*Commonwealth Scientific and Industrial Research Organization, Sydney, Australia
and University College, London*

1. INTRODUCTION

A number of recent papers have discussed the use in statistical analysis of the 'Studentized' range, that is to say, the ratio of (a) the range in a sample of n observations from a population having standard deviation σ to (b) an *independent* root-mean-square estimate of σ based on ν degrees of freedom. Tables of the upper 5 and 1% points of this ratio, in the case of normal variation, have been given by Pearson & Hartley (1954, Table 29). In the following paper we are concerned with a different statistic, namely, the ratio of the range, w , to the standard deviation, s , both measures of variation being calculated on the same sample of n observations. While the first ratio can play a useful part in providing certain rapid tests in the analysis of variance, the second, whose value depends only on the configuration of a particular sample, may be useful (with suitable tables) in detecting heterogeneity of data or departure from normality.

Our interest in this matter arose as a result of correspondence between one of us and Dr Joseph Berkson who has for some time carried out a routine scrutiny of data by making a comparison of the range and standard deviation estimators of σ ; in this connexion he initiated an empirical investigation into the correlation between the estimators in order to determine the standard error of the difference between them. Some years earlier, however, G. A. Baker (1946) suggested the use of the ratio w/s as a means of detecting lack of homogeneity, and showed by an artificial sampling experiment that the distribution of this ratio might be expected to change considerably with the form of the parent population.

The object of the present paper is to provide tables of certain upper and lower percentage points of $u = w/s$ for samples of n observations from a single normal population. Two methods of attack will be used:

(a) We shall show that the exact moments of the distribution of u can be derived quite simply from the known moments of s and w . Hence approximations to the percentage points may be obtained from representing the distribution by any suitable frequency curve having the same moments, e.g. by a curve of the Pearson system.

(b) Using a method employed by Pearson & Chandra Sekar (1936), we shall show how in small samples values of the upper percentage points of u may be calculated exactly.

In an overlapping region of the table, these two methods provide some confirmation of the accuracy of the results.

Finally, we shall give some numerical illustrations of how the ratio might be used to provide a quick assessment of the homogeneity or normality of data.

2. METHOD USING THE MOMENTS OF THE DISTRIBUTION OF $u = w/s$

Let x_1, x_2, \dots, x_n represent a sample of n independent observations from a normal population having variance σ^2 . Write

$$\bar{x} = \sum_{i=1}^n x_i/n, \quad s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/(n-1), \quad w = x_{\max.} - x_{\min.}$$

It may be shown, following an argument similar to that used by Geary (1933, p. 185; 1936, p. 296) and since employed by Kamat (1953, p. 124), that the joint distribution of the ratios $(x_i - \bar{x})/s$ ($i = 1, 2, \dots, n$) is independent of s . Hence any function of these ratios and in particular the difference of $(x_{\max.} - \bar{x})/s$ and $(x_{\min.} - \bar{x})/s$, or w/s , will be independent of s . From this it follows that

$$\mathcal{E}(w^r) = \mathcal{E}(w/s)^r = \mathcal{E}(w^r)/\mathcal{E}(s^r). \tag{1}$$

On this basis the moments of $u = w/s$ may be computed, making use of the moments of w as far as they have been determined, and the fully known moments of s . We shall be concerned here only with the moments of w up to the fourth; for $n \leq 20$ these have been given by Hartley & Pearson (1951) while for $n \geq 20$ values of the standard deviation and the β_1, β_2 coefficients were computed by Tippett (1925) (with some later modification by Pearson (1926)) at $n = 20, 60, 100, 200, 500$ and 1000 . $\mathcal{E}(w)$ was, of course, given by Tippett for all sample sizes up to 1000 .

The procedure was then as follows:

(a) Taking as framework the sample sizes $n = 10, 15, 20, 60, 100, 200, 500, 1000$, values of $\mathcal{E}(w^r)$ for $r = 2, 3, 4$ were found from the moments about the mean.

(b) The corresponding expectations of s^r were obtained from

$$\mathcal{E}(s^r) = \sigma^r (2/(n-1))^{r/2} \Gamma\{\frac{1}{2}(n-1+r)\} / \Gamma\{\frac{1}{2}(n-1)\}, \tag{2}$$

and hence $\mathcal{E}(w^r)$ was determined from (1).

(c) The 2nd, 3rd and 4th moments of u about zero were then transferred to the mean, $\bar{u} = \mathcal{E}(u) = \mathcal{E}(w)/\mathcal{E}(s)$, and finally values of $\sigma_u, \beta_1(u)$ and $\beta_2(u)$ obtained. With the values for the moments of w at present available for large n , σ_u could only be calculated to three and the beta coefficients to two significant figures.

(d) Pearson curves with the same first four moments were then used to approximate to upper and lower $100\alpha\%$ points for $\alpha = 0.10, 0.05, 0.025, 0.01$ and 0.005 and for the framework values of n quoted in (a) above.

It was found that if these percentage points were expressed as standardized deviates, $U_\alpha = (u_\alpha - \bar{u})/\sigma_u$, then the resulting values of U_α were sufficiently smooth for interpolation in the panels of the frame for n . Using these interpolated values and the formula

$$u_\alpha = \bar{u} + U_\alpha \sigma_u,$$

where \bar{u} is known to four and σ_u to three decimal places,* the panels were filled in to give additional percentage points for u at $n = 12, 30, 40, 50, 80$ and 150 .

For $n \leq 20$ the percentage points should be accurate to within ± 0.02 and a comparison with the exact percentage points (see § 3, Table 1), where these are available, does not reveal discrepancies beyond 0.01 . For $n > 20$ the uncertainty in the beta coefficients of w alone may result in an uncertainty of ± 0.02 in the percentage points.

* Apart from Tippett's (1925) values of σ_w required for σ_u , additional values at $n = 30, 45$ and 75 have been given by Pearson (1932, p. 405).

3. DERIVATION OF THE EXACT VALUES OF THE UPPER PERCENTAGE POINTS IN SMALL SAMPLES

We have here followed a method previously used by Pearson & Chandra Sekar (1936) when determining upper percentage points of the distribution of $(x_{\max} - \bar{x})/s$ in sampling from a normal population. This method proceeds as follows.

Suppose that the observations x_i ($i = 1, 2, \dots, n$) are arranged in some random order, say the order in which they are drawn. Then we may select a particular pair, say the j th and k th drawn, and form the ratio

$$u' = (x_j - x_k)/s, \tag{3}$$

where s is the sample standard deviation defined above. It is then easy to show that the distribution of u' is functionally related to that of Student's t . This follows because the total sum of squares of deviations from the sample mean, $(n - 1) s^2$, contains the single degree of freedom component $\frac{1}{2}(x_j - x_k)^2$, so that

$$(n - 1) s^2 = \frac{1}{2}(x_j - x_k)^2 + \chi^2 \sigma^2,$$

χ^2 being based on $n - 2$ degrees of freedom and being independent of $\frac{1}{2}(x_j - x_k)^2$. In fact, we can write in more detail

$$\chi^2 \sigma^2 = \sum_{i \neq j, k} (x_i - \bar{x}')^2 + 2(n - 2) (\bar{x}' - \bar{x}'')^2/n,$$

with

$$\bar{x}' = \sum_{i \neq j, k} x_i / (n - 2), \quad \bar{x}'' = \frac{1}{2}(x_j + x_k).$$

It follows that

$$u' = \frac{(x_j - x_k) \sqrt{(n - 1)}}{\{\chi^2 \sigma^2 + \frac{1}{2}(x_j - x_k)^2\}^{\frac{1}{2}}} = \frac{t \sqrt{\{2(n - 1)\}}}{\{t^2 + n - 2\}^{\frac{1}{2}}}, \tag{4}$$

where t is based on $n - 2$ degrees of freedom. If now we consider a particular sample, there will be $n(n - 1)$ different combinations of j and k , each providing a value of u' . These $n(n - 1)$ values may be arranged in descending order of magnitude and denoted by

$$u'_{[1]} \geq u'_{[2]} \geq \dots \geq u'_{[n(n-1)]}.$$

Clearly the distribution of u' , determined from (4), for a pair j, k of x 's taken at random will be formed from the sum of the $n(n - 1)$ equally weighted distributions of $u'_{[1]}, \dots, u'_{[n(n-1)]}$. The distribution of $u = w/s$ is the distribution of $u'_{[1]}$. In so far as there is no overlap of the distributions of $u'_{[1]}$ and $u'_{[2]}$ (and consequently of $u'_{[1]}$ and $u'_{[3]}$, etc.), the shape of the distribution of $u = u'_{[1]}$ will be that of the upper tail of the known distribution of u' .

As shown in the Appendix, the maximum value of $u'_{[2]}$ is $\{\frac{3}{2}(n - 1)\}^{\frac{1}{2}}$; this arises when the configuration of the sample is as follows:



For n of moderate size, such a configuration with $n - 3$ observations having the same or very nearly the same value is most unlikely to occur, so that even when $u < \{\frac{3}{2}(n - 1)\}^{\frac{1}{2}}$ the overlap of the distributions of $u'_{[2]}$ and $u = u'_{[1]}$ may be of little importance.

Formally, we may say that for any positive quantity $U \geq \{\frac{3}{2}(n-1)\}^\ddagger$,

$$\begin{aligned} \Pr\{u' \geq U\} &= \Pr\{u' \text{ is } u_{[1]}\} \times \Pr\{u'_{[1]} = u \geq U\} \\ &= \frac{1}{n(n-1)} \times \Pr\{u \geq U\} \end{aligned}$$

or
$$\Pr\{u \geq U\} = n(n-1) \Pr\{u' \geq U\}. \tag{5}$$

This means that provided $U \geq \{\frac{3}{2}(n-1)\}^\ddagger$, the upper tail area of the distribution of $u = w/s$ can be obtained exactly from that of u' which, through the relation (4), can be derived from the t -distribution. Alternatively, we can compute the $100\alpha\%$ point of u (say $u(\alpha, n)$) from the $100\alpha/\{n(n-1)\}\%$ point of t for $n-2$ degrees of freedom, using the formula

$$u^2(\alpha, n) = 2(n-1) t^2(\alpha', \nu) / \{\nu + t^2(\alpha', \nu)\}, \tag{6}$$

where $\alpha' = \alpha/\{n(n-1)\}$ and $\nu = n-2$.

The following table compares at $n = 10, 15$ and 20 the upper percentage points of $u = w/s$ obtained by this method and by that of the preceding section, for $\alpha = 0.10, 0.05, 0.025, 0.01$ and 0.005 . It also gives the limiting values $\{\frac{3}{2}(n-1)\}^\ddagger$. It will be seen that at $n = 10$ four of the percentage points lie beyond the upper limit for $u'_{[2]}$, but that for $n = 15$ and 20 all the percentage points lie below this limit. Thus, strictly speaking, the second method of determining $u(\alpha, n)$ is not applicable for much of the table beyond $n = 10$. In fact, however, we find surprising agreement between the Pearson curve values and those derived from u' right up to $n = 20$, suggesting that the probability of $u'_{[2]}$ falling in the overlapping upper tail area of its distribution must be very small. As would be expected, the difference between the two results, where it appears, makes the limit obtained from the u' -distribution slightly larger than that based on moments.

Table 1. Comparison of percentage points for $u = w/s$ obtained by different methods

n	Method of derivation	Percentage points, $u(\alpha, n)$					Max. value of $u'_{[2]}$
		$\alpha = 0.10$	0.05	0.025	0.01	0.005	
10	From curve fitted by moments	3.57	3.69	3.78	3.88	3.94	—
	From u' -distribution	3.574	3.685-	3.777	3.875-	3.935-	3.674
15	From curve fitted by moments	4.02	4.17	4.29	4.43	4.52	—
	From u' -distribution	4.034	4.173	4.295+	4.435+	4.527	4.583
20	From curve fitted by moments	4.32	4.49	4.63	4.79	4.91	—
	From u' -distribution	4.342	4.496	4.635+	4.799	4.910	5.339

4. COMPOSITION OF THE FINAL TABLE

The results following from the methods of the two preceding sections have been put together in Table 6 printed at the end of the paper (p. 491 below).

For the lower percentage points, the lines corresponding to $n = 10, 15, 20, 60, 100, 200, 500$ and 1000 have been computed by the method of § 2, i.e. from the fitted Pearson curves,

and the remaining lines by interpolation. The same applies to the upper percentage points for $n \geq 20$. For $n < 20$, the values shown to three decimal accuracy are exact, being in every case less than the limit $\{\frac{3}{2}(n-1)\}^{\frac{1}{2}}$, and have been computed from formula (6). For the remaining upper percentage points with $n \leq 20$, the values yielded by formula (6) were reduced slightly, the reduction amounting to either 1 or 2 units in the second decimal. This reduction was assessed by smoothing the discrepancies between the two methods shown in Table 1.

We have not attempted to interpolate for $n > 200$. A rough interpolation can easily be made by graphical or other methods, but if more accurate values of percentage points of w/s for such large sample sizes were really required, it would seem necessary to compute additional values of the higher moments of the range between Tippett's values at $n = 200$, 500 and 1000.

5. ILLUSTRATIONS OF POSSIBLE USES OF THE TABLE OF PERCENTAGE POINTS

We have pressed forward with the preparation of this table, involving certain inaccuracy for the larger samples, because the practical usefulness of the ratio w/s can only be tested with a table available. The following examples illustrate some of a variety of ways in which the table might be used, but it must be emphasized that no claim is made that the test is better than other simple tests which have been or could be devised.

(5.1) *Testing for lack of homogeneity of data*

As mentioned above Dr Joseph Berkson has made it a practice in course of consultation work with laboratory colleagues, to compare standard deviations derived from samples of 50–200 observations with the range estimate of σ found by dividing the sample range by factor d_n obtained from Tippett's (1925) table. He writes:

'On occasion there was a considerable difference between the range and mean-square estimates and in these instances, almost invariably, further investigation disclosed that either there had been an arithmetic error, or the observations were very different from normally distributed; in particular there were one or more observations outlying from the rest. This sort of experience, accumulated over a long period of time, built up the impression that the range estimate of the standard deviation of a normal distribution was a close equivalent of the mean-square estimate.'

To illustrate the use of our table in this connexion we have considered some of the frequency distributions of physical characters published by Mahalanobis, Majumdar & Rao (1949) in the report on an anthropometric survey of the United Provinces of India undertaken in 1941. From these data we have taken the frequency distributions of head length given for each of 23 racial groups. (Appendix 7 of the paper and also Tables 2.1 and 6.1.) The sample size varies from 57 to 197; for each of the 23 groups we have calculated the ratio w/s and the results are plotted in Fig. 1 on which are also drawn the upper and lower 5 and 1% significance levels for the ratio. For these levels, smooth curves have been drawn through points taken from Table 6. It will be seen that for five of the twenty-three groups w/s falls beyond the upper 5% level and for two of these, well beyond the 1% level. On looking at the original frequency distributions, it was found that in each of these five cases there was one or more than one very divergent measurement, always in the sense of a small head length. Table 2 gives some idea of the position.

The notation x_I has been used to denote the lowest observation in the sample and x_{II} the lowest but one, except in Group 12 where it represents the lowest but two. The ratios $(x_I - \bar{x})/s$ and $(x_I - x_{II})/s$ give some idea of the divergence of the outliers.

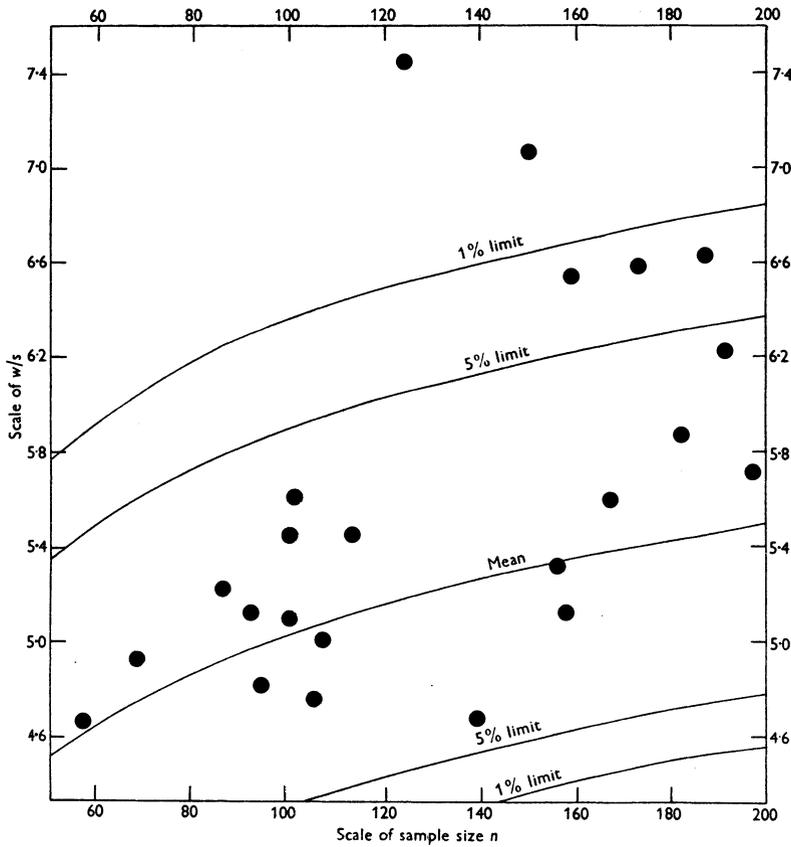


Fig. 1 Head length data for 23 Indian racial groups.

Table 2. Data from five groups showing abnormal w/s (the measurements of head length are in millimetres)

Group no.	n	\bar{x}	s	w	w/s	$(x_I - \bar{x})/s$	$(x_I - x_{II})/s$	b_2
6	150	186.10	6.94	49	7.1	-5.2	-2.0	6.82
7	124	186.94	6.04	45	7.5	-4.6	-2.6	5.61
8	187	181.87	5.88	39	6.6	-4.1	-1.2	4.30
12	173	187.69	6.69	44	6.6	-3.5 (2)*	-1.2	4.46
15	159	186.85	6.42	42	6.5	-3.7	-1.4	3.72

* Two observations fell at the extreme value of 164 mm.

Without inside knowledge of the investigation we cannot, of course, judge whether any of these outlying readings are due to faulty recording, but had a test of this character been applied to the distributions originally there would have been a *prima facie* case for further scrutiny of the data before, for example, calculating higher moments.

The Indian Report gives also the values of the shape coefficients $\sqrt{b_1}$ and b_2 for each of the twenty-three distributions. Taking b_2 as a measure of kurtosis most likely to be related to w/s and referring to a table of significance levels to test for departure from normality (see Pearson, 1930; Pearson & Hartley, 1954, Tables 34C), it is found that the b_2 values for groups 6, 7, 8 and 12 are significant at the upper 1% level and that for group 15 at the upper 5% level. Further, none of the other 18 values of b_2 differ significantly at 5% level from the normal value of $\beta_2 = 3.0$!

In the present case it would seem probable that departure from normality is due to the presence of discrepant observations; under these circumstances there is little reason to calculate $\sqrt{b_1}$ and b_2 which are only useful as measures of shape when dealing with homogeneous non-normal material. However, the similarity in behaviour of w/s and b_2 in the present example, suggests that under suitable conditions the former ratio may provide a useful and easily calculated measure of kurtosis. An example of this application will be considered next.

(5.2) Testing for kurtosis

Geary (1935) and Pearson (1935) in a joint investigation used some artificial sampling data to compare the efficiency of b_2 and m/s (i.e. Geary's (mean deviation)/(standard deviation) ratio) in detecting departure from normality. We have now calculated the w/s ratio on the same data. These consist of samples of 76 and 300 from the following symmetrical populations:

Population no.	Type	β_2
I	Rectangular	1.8
II	Pearson Type II	2.5
III	Pearson Type VII	4.1
IV	Pearson Type VII	7.1
V	Double exponential	6.0

Table 3. Samples of 76; comparison of test ratios

Population ...	I (rectangular)			IV (Type VII, $\beta_2 = 7.1$)			V (double exponential)		
	m/s	b_2	w/s	m/s	b_2	w/s	m/s	b_2	w/s
Number of sample points:									
Within 5% limits	0	0	0	3	2	7	1	1	3
Between 5 and 1% limits	2	0	0	1	4	1	1	5	3
Beyond 1% limits	8	10	10	6	4	2	8	4	4

A comparison of the b_2 and w/s values for ten samples with $n = 76$ from populations nos. I, IV and V and for six samples with $n = 300$ from populations II and III is made in Figs. 2 and 3. Similar diagrammatic comparisons between b_2 and Geary's ratio m/s were given in Pearson's (1935), Figs. 3 and 4. The high correlation between b_2 and w/s will be noted, particularly for the platykurtic distributions I and II. The relative power of the three ratios in testing for departure from normality is summarized in Tables 3 and 4 as far as these very limited sampling results are concerned. It will be seen that the w/s ratio is in some cases as effective as the other two statistics, but is never more effective than b_2 . On

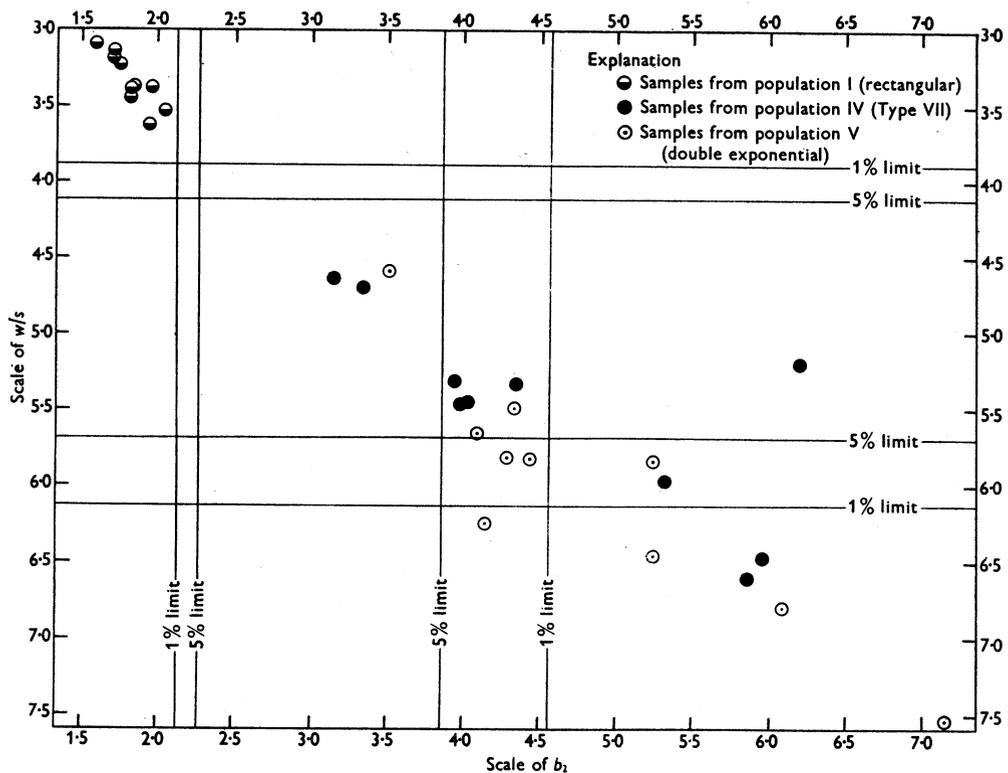


Fig. 2. Comparison of test ratios for samples of 76.

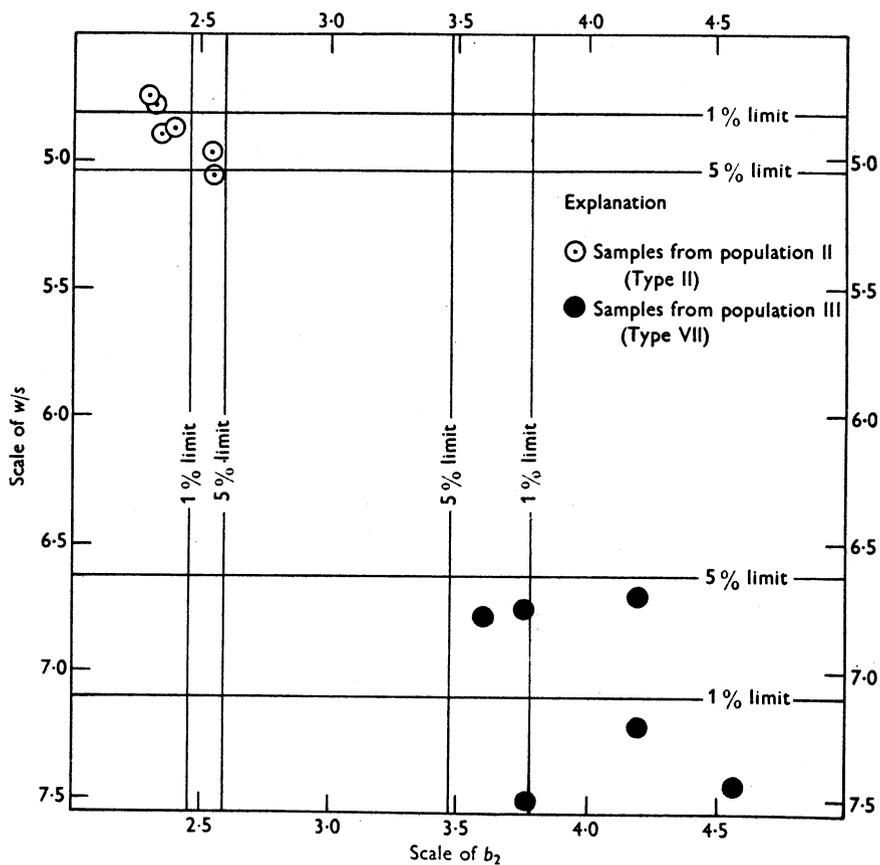


Fig. 3. Comparison of test ratios for samples of 300.

this evidence, all that we would suggest is that the use of this very simply computed criterion deserves further examination in practice.

It should be noted, as pointed out by Geary and Pearson that the position of the percentage points for b_2 with $n = 76$ were not then and have not since been firmly established.

Table 4. *Samples of 300; comparison of test ratios*

Population ...	II (Type II, $\beta_2 = 2.5$)			III (Type VII, $\beta_2 = 4.1$)		
	<i>m/s</i>	b_2	<i>w/s</i>	<i>m/s</i>	b_2	<i>w/s</i>
Number of sample points:						
Within 5% limits	2	0	1	2	0	0
Between 5 and 1% limits	0	2	3	2	3	3
Beyond 1% limits	4	4	2	2	3	3

(5.3) *A small sample illustration*

Only experience can show whether the table of limits for *w/s* will prove useful in detecting abnormality in the configuration of small samples. We give here a single example illustrating a situation which might arise where some of the observations in a sample are subject to greater variability than the rest, but there is no alteration in mean value.

Samples of 20 values were taken from Wold's (1948) *Table of Random Normal Deviates*. In each sample the first 16 values were left unaltered and the remaining four were multiplied (a) by three and (b) by four. Thus we have composite samples of $n = 20$, each containing 16 observations from a population $N(0, \sigma)$ and four from $N(0, k\sigma)$ with $\sigma = 1$ and (a) $k = 3$, (b) $k = 4$. How often will the ratio of range to standard deviation suggest the presence of this heterogeneity? The procedure was applied to the first five series of 20 numbers in Wold's table, i.e. to the numbers in the first two columns of his p. 2, 1, 1-20; 1, 21-40; 1, 41-2, 10; 2, 11-30; 2, 31-50. The results are shown in Table 5, the significance levels for $n = 20$ being found from Table 6.

Table 5. *Test for lack of constancy in standard deviation*

Sample	With $k = 3$		With $k = 4$	
	<i>w/s</i>	Significant at	<i>w/s</i>	Significant at
1	4.82	1 % level	4.93	0.5 % level
2	4.70	2.5 % level	4.74	2.5 % level
3	3.75	—	4.26	—
4	3.98	—	4.06	—
5	3.61	—	3.45	—

Evidence of something anomalous appeared in the first two samples, but for the remainder *w/s* was not even significant at the 10% level (single tail test). It is clear that if all four observations from the population with larger variance happen to lie near the population mean, a very considerable multiplying up of the standard deviation may be required before

any of them becomes an extreme observation in the composite sample. In fact, the range may remain stationary and the standard deviation increase until k has become quite large. This is the position for the fifth sample in the series.

If now we combine the five samples into a single sample of $n = 100$, containing 80 observations from $N(0, \sigma)$ and 20 from $N(0, k\sigma)$, we find

$$\text{for } k = 3, \quad w/s = 7.95; \quad \text{for } k = 4, \quad w/s = 8.72,$$

results which both lie well beyond the 0.5% level. With as many as 20 observations from the more variable population, it becomes almost certain that there will be some values appearing as outliers in the tails of the distribution.

Table 6. *Percentage points of the distribution of the ratio of range to standard deviation, w/s , in samples of size n from a normal population*

Size of sample n	Lower percentage points					Upper percentage points					Size of sample n
	0.5	1.0	2.5	5.0	10.0	10.0	5.0	2.5	1.0	0.5	
						1.997	1.999	2.000	2.000	2.000	3
						2.409	2.429	2.439	2.445	2.447	4
						2.712	2.753	2.782	2.803	2.813	5
						2.949	3.012	3.056	3.095	3.115	6
						3.143	3.222	3.282	3.338	3.369	7
						3.308	3.399	3.471	3.543	3.585	8
						3.449	3.552	3.634	3.720	3.772	9
10	2.47	2.51	2.59	2.67	2.77	3.57	3.685	3.777	3.875	3.935	10
11	2.53	2.58	2.66	2.74	2.84	3.68	3.80	3.903	4.012	4.079	11
12	2.59	2.65	2.73	2.80	2.91	3.78	3.91	4.01	4.134	4.208	12
13	2.65	2.70	2.78	2.86	2.97	3.87	4.00	4.11	4.244	4.325	13
14	2.70	2.75	2.83	2.91	3.02	3.95	4.09	4.21	4.34	4.431	14
15	2.75	2.80	2.88	2.96	3.07	4.02	4.17	4.29	4.43	4.53	15
16	2.80	2.85	2.93	3.01	3.13	4.09	4.24	4.37	4.51	4.62	16
17	2.84	2.90	2.98	3.06	3.17	4.15	4.31	4.44	4.59	4.69	17
18	2.88	2.94	3.02	3.10	3.21	4.21	4.38	4.51	4.66	4.77	18
19	2.92	2.98	3.06	3.14	3.25	4.27	4.43	4.57	4.73	4.84	19
20	2.95	3.01	3.10	3.18	3.29	4.32	4.49	4.63	4.79	4.91	20
30	3.22	3.27	3.37	3.46	3.58	4.70	4.89	5.06	5.25	5.39	30
40	3.41	3.46	3.57	3.66	3.79	4.96	5.15	5.34	5.54	5.69	40
50	3.57	3.61	3.72	3.82	3.94	5.15	5.35	5.54	5.77	5.91	50
60	3.69	3.74	3.85	3.95	4.07	5.29	5.50	5.70	5.93	6.09	60
80	3.88	3.93	4.05	4.15	4.27	5.51	5.73	5.93	6.18	6.35	80
100	4.02	4.09	4.20	4.31	4.44	5.68	5.90	6.11	6.36	6.54	100
150	4.30	4.36	4.47	4.59	4.72	5.96	6.18	6.39	6.64	6.84	150
200	4.50	4.56	4.67	4.78	4.90	6.15	6.38	6.59	6.85	7.03	200
500	5.06	5.13	5.25	5.37	5.49	6.72	6.94	7.15	7.42	7.60	500
1000	5.50	5.57	5.68	5.79	5.92	7.11	7.33	7.54	7.80	7.99	1000

APPENDIX

Determination of the maximum value of $u'_{[2]}$

Let x_i ($i = 1, 2, \dots, n$) denote the values of a random sample of n observations arranged in ascending order of magnitude, so that x_1 is the smallest and x_n the largest.* We wish to determine the maximum value of the ratio $u'_{[2]} = (x_{n-1} - x_1)/s$, where $s^2 = \sum_i (x_i - \bar{x})^2 / (n - 1)$.

Since the value will not depend on the origin or scale of the x_i we may assume, without loss of generality, that $x_1 = 0$ and $x_{n-1} = 1$, so that $x_{n-1} - x_1 = 1$ and the maximum of $u'_{[2]}$ can be found by minimizing the sum of squares

$$S = (n - 1) s^2 = \sum x_i^2 - (\sum x_i)^2 / n.$$

This minimum will be found in three stages:

(1) With $x_1 = 0, x_{n-1} = 1$ and x_n held fixed, S is minimized if all the x_i for $i = 2, 3, \dots, n - 2$ have the same value, say X . For if $X = \sum_{i=2}^{n-2} x_i / (n - 3)$, we may write

$$S = 1 + x_n^2 + \sum_{i=2}^{n-2} (x_i - X)^2 + (n - 3) X^2 - (1 + x_n + (n - 3) X)^2 / n,$$

which is clearly minimized if all $x_i = X$ ($i = 2, 3, \dots, n - 2$).

(2) Confining ourselves to samples in which $x_1 = 0, x_i = X$ ($i = 2, 3, \dots, n - 2$) and $x_{n-1} = 1$, we may next show that S is minimized if $x_n = 1$. We now have

$$S = 1 + x_n^2 + (n - 3) X^2 - (1 + x_n + (n - 3) X)^2 / n.$$

Then

$$\frac{\partial S}{\partial x_n} = 2(x_n - \bar{x}) > 0,$$

so that the smallest possible value of x_n , namely, $x_n = x_{n-1} = 1$, will minimize S .

(3) Next, confining ourselves to samples with

$$x_1 = 0, \quad x_n = x_{n-1} = 1 \quad \text{and} \quad x_i = X \quad (i = 2, 3, \dots, n - 2),$$

we must find the value of X which minimizes S . Clearly now

$$S = 2 + (n - 3) X^2 - (2 + (n - 3) X)^2 / n,$$

an expression which is easily found to have a minimum at $X = \frac{2}{3}$. For this we have

$$S = 2 + \frac{4}{9}(n - 3) - (2 + (n - 3)\frac{2}{3})^2 / n = \frac{2}{3}.$$

Hence

$$s^2 = S / (n - 1) = 2 / \{3(n - 1)\},$$

and the maximum value of $u'_{[2]}$ is $\{\frac{3}{2}(n - 1)\}^{\frac{1}{2}}$.

For this configuration maximizing $u'_{[2]}$ we have a single observation at one extreme of the sample range, two equal observations at the other extreme and the remaining $n - 3$ observations together at the mean of the whole, i.e. at a distance two-thirds of the way between the first extreme and the second extreme.

* Note the change from § 3, where the observations were taken in random order.

REFERENCES

- BAKER, G. A. (1946). *Ann. Math. Statist.* **17**, 366.
GEARY, R. C. (1933). *Biometrika*, **25**, 184.
GEARY, R. C. (1935). *Biometrika*, **27**, 310, 353.
GEARY, R. C. (1936). *Biometrika*, **28**, 295.
HARTLEY, H. O. & PEARSON, E. S. (1951). *Biometrika*, **38**, 463.
KAMAT, A. R. (1953). *Biometrika*, **40**, 116.
MAHALANOBIS, P. C., MAJUMDAR, D. N. & RAO, C. R. (1949). *Sankhyā*, **9**, 89.
PEARSON, E. S. (1926). *Biometrika*, **18**, 173.
PEARSON, E. S. (1930). *Biometrika*, **22**, 239.
PEARSON, E. S. (1932). *Biometrika*, **24**, 404.
PEARSON, E. S. (1935). *Biometrika*, **27**, 333.
PEARSON, E. S. & CHANDRA SEKAR, C. (1936). *Biometrika*, **28**, 308.
PEARSON, E. S. & HARTLEY, H. O. (1954). *Biometrika Tables for Statisticians*, **1**. Cambridge University Press.
TIPPETT, L. H. C. (1925). *Biometrika*, **17**, 364.
WOLD, H. (1948). *Random Normal Deviates*. Tracts for Computers, no. 25. Cambridge University Press.