1.5 - 1.8 Homework Problems

(5.1) Use properties of conjugates and moduli to show the equality
(1a) $\overline{z+3i} = \bar{\bar{z}} + \overline{3i} = z - 3i$

(1b) $\overline{iz} = \bar{i}\bar{z} = -i\bar{z}$

(1c) $\overline{(2+i)^2} = \overline{(2+i)(2+i)} = \overline{(2+i)} \quad \overline{(2+i)} = (2-i)(2-i) = 4 - 1 - 2i - 2i = 3 - 4i$

(5.2) Sketch the set of points determined by $\operatorname{Re}(\bar{z} - i) = 2$.

First, let's write $\bar{z} - i$ in terms of x and y.

$$\bar{z} - i = \overline{x+iy} - i = x - iy - i = x + i(-y-1)$$

So $re(\bar{z} - i) = x$. The set of points we need to draw is a vertical line at $x = 2$.

(5.3) Verify the two properties of complex conjugates

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\overline{z_1 z_2} &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\
&= (x_1x_2 - (-y_1)(-y_2)) + i(x_1(-y_2) + x_2(-y_1)) \\
&= (x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1\bar{z}_2
\end{aligned}$$

$$\begin{aligned}
\overline{\frac{z_1}{z_2}} &= \overline{\frac{x_1 + iy_1}{x_2 + iy_2}} \\
&= \overline{\frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2}} \\
&= \overline{\frac{x_1x_2 + y_1y_2 + iy_1x_2 - ix_1y_2}{x_2^2 + y_2^2}} \\
&= \frac{x_1x_2 - y_1y_2 - iy_1x_2 + ix_1y_2}{x_2^2 + y_2^2} \\
\frac{\bar{z}_1}{\bar{z}_2} &= \frac{x_1 - iy_1}{x_2 - iy_2} \\
&= \frac{x_1 - iy_1}{x_2 - iy_2}\frac{x_2 + iy_2}{x_2 + iy_2} \\
&= \frac{x_1x_2 + y_1y_2 - iy_1x_2 + ix_1y_2}{x_2^2 + y_2^2} = \overline{\frac{z_1}{z_2}}
\end{aligned}$$

(5.7) Show that $|\operatorname{Re}(2 + \bar{z} + z^3)| \leq 4$ when $|z| \leq 1$.

Since $|\text{Re}\, z| \leq |z|$ for every z, we know that

$$|\text{Re}\,(2 + \bar{z} + z^3)| \leq |2 + \bar{z} + z^3|$$

Then apply the Triangle Inequality twice to put the modulus on the separate terms.

$$|2 + \bar{z} + z^3| \leq |2| + |\bar{z}| + |z^3|$$

Since $|\bar{z}| = |z|$ and $|z^3| = |z|^3$,

$$|2| + |\bar{z}| + |z^3| = 2 + |z| + |z|^3$$

Now plug in $|z| \leq 1$

$$2 + |z| + |z|^3 \leq 2 + 1 + 1^3 = 4$$

Together these inequalities give the desired inequality.

(5.9) Show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

$$\begin{aligned}
|z^4 - 4z^2 + 3| &= |(z^2 - 3)(z^2 - 1)| \\
&= |z^2 - 3||z^2 - 1| \\
&\geq \big||z^2| - |3|\big| \, \big||z^2| - |1|\big| \quad \text{backwards Tri. Ineq. on both} \\
&= \big||z|^2 - 3\big| \, \big||z|^2 - 1\big| \\
&= |2^2 - 3||2^2 - 1| \\
&= |4 - 3||4 - 1| \\
&= 3
\end{aligned}$$

So $|z^4 - 4z^2 + 3| \geq 3$. By taking reciprocals (which reverses the inequality) we get the desired inequality.

(5.10a) Prove that z is real if and only if $\bar{z} = z$.

($\Rightarrow$) Assume z is real. Then $z = x + i0 = x$ and $\bar{z} = x - i0 = x$. So $\bar{z} = z$.

($\Leftarrow$) Assume $\bar{z} = z$. Then $x + iy = x - iy$, which is only true if $x = x$ and $y = -y$. $y = -y$ means $2y = 0$ hence $y = 0$, so z is real.

(2a) $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$

(2b) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

(8.4) Using the fact that $|e^{i\theta} - 1|$ gives the distance between $e^{i\theta}$ and 1, give a geometric argument to find a value of θ in the interval $0 \leq \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

We know that $|e^{i\theta} - 1|$ gives the distance between $e^{i\theta}$ and 1. We also know that $e^{i\theta}$ always lies on the unit circle. The only point on the unit circle which is a distance 2 from 1 is -1. If $e^{i\theta} = -1$ and $0 \le \theta < 2\pi$, then $\theta = \pi$.

(8.5a) Show the following equality by switching to exponential form.

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+i\sqrt{3})$$

$$\begin{aligned} i(1-\sqrt{3}i)(\sqrt{3}+i) &= e^{\frac{i\pi}{2}}(2e^{-\frac{i\pi}{3}})(2e^{\frac{i\pi}{6}}) \\ &= 4e^{i(\frac{3\pi}{6}-\frac{2\pi}{6}+\frac{\pi}{6})} \\ &= 4e^{\frac{i2\pi}{6}} \\ &= 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \\ &= 2(1+i\sqrt{3}) \end{aligned}$$

(8.5b) Show the following equality by switching to exponential form.

$$(-1+i)^7 = 8(-1-i)$$

$$\begin{aligned} (-1+i)^7 &= (\sqrt{2}e^{\frac{i3\pi}{4}})^7 \\ &= 2^{\frac{7}{2}}e^{\frac{i21\pi}{4}} \\ &= 2^3 2^{\frac{1}{2}} e^{i4\pi} e^{\frac{i5\pi}{4}} \\ &= 8(\sqrt{2})(1)(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) \\ &= 8(-1-i) \end{aligned}$$

(8.6) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ (remember Arg refers to the principal argument).

Assume $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Both are in the right half of the plane, so $-\frac{\pi}{2} < \operatorname{Arg} z_1 < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \operatorname{Arg} z_2 < \frac{\pi}{2}$. We know that when we multiply $z_1 z_2$, we add the arguments. Normally, we wouldn't know that $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is the principal argument $\operatorname{Arg} z_1 z_2$, since the sum might not be between $-\pi$ and π. However, with the right half plane assumption both $\operatorname{Arg} z_1$ and $\operatorname{Arg} z_2$ are less than $\frac{\pi}{2}$ so their sum is less than π, and both are bigger than $-\frac{\pi}{2}$ so their sum is bigger than π. Therefore $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

(8.10) Use de Moivre's formula to derive the triple angle formulas.

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

Plug $n = 3$ into deMoivre's formula. Use the real parts for (10a) and the imaginary parts for (10b)

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= \cos n\theta + i\sin n\theta \\
(\cos\theta + i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta \quad AND \quad & 3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta
\end{aligned}$$