10 Problems

(1a) $(2i)^{\frac{1}{2}} = (2e^{i(\frac{\pi}{2}+2\pi k)})^{\frac{1}{2}} = \sqrt{2}(e^{i(\frac{\pi}{4}+\pi k)})$

$e^{i\pi k}$ is either -1 or 1, and $\sqrt{2}e^{i\frac{\pi}{4}} = 1+i$. So the roots are $1+i$ and $-1-i$.

(1b) $(1-\sqrt{3}i)^{\frac{1}{2}} = (2e^{i(\frac{-\pi}{3}+2\pi k)})^{\frac{1}{2}} = \sqrt{2}e^{i(\frac{-\pi}{6}+\pi k)}$

Again $e^{i\pi k} = \pm 1$. $e^{i\frac{-\pi}{6}} = \frac{\sqrt{3}-i}{2}$, so the roots are $\frac{\sqrt{3}-i}{\sqrt{2}}$ and $\frac{-\sqrt{3}+i}{\sqrt{2}}$.

(2a) $(-16)^{\frac{1}{4}} = (16e^{i(\pi+2\pi k)})^{\frac{1}{4}} = 2e^{i(\frac{\pi}{4}+\frac{\pi k}{2})}$

$e^{i\frac{\pi k}{2}}$ is either 1, i, -1 or $-i$. $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$. So the roots are $\pm\sqrt{2}(1+i)$ and $\pm\sqrt{2}(1-i)$. The principal root is $\sqrt{2}(1+i)$ (when $k=0$).

(2b) $(-8-8\sqrt{3}i)^{\frac{1}{4}} = (8\cdot 2e^{i(\frac{-2\pi}{3}+2\pi k)})^{\frac{1}{4}} = 2e^{i(\frac{-\pi}{6}+\frac{\pi k}{2})}$

Again $e^{i\frac{\pi k}{2}}$ is ± 1 or $\pm i$. $e^{-i\frac{\pi}{6}} = \frac{\sqrt{3}-i}{2}$, so the roots are $\pm(\sqrt{3}-i)$ and $\pm(1-\sqrt{3}i)$. The principal root is $\sqrt{3}-i$ (when $k=0$).

(6)

$$\begin{aligned} z^4+4 &= 0 \\ z^4 &= -4 \\ z^4 &= 4e^{i(\pi+2\pi k)} \\ z &= \sqrt{2}e^{i(\frac{\pi}{4}+\frac{\pi k}{2})} \\ z &= \sqrt{2}(\frac{1+i}{\sqrt{2}})(e^{i\frac{\pi k}{2}}) \\ z &= \pm(1+i)\, OR \pm(1-i) \end{aligned}$$

Note: the book version of this problem has another part, where you factor z^4+4. That part is done as follows:

$$\begin{aligned} z^4+4 &= (z-(1+i))(z-(-1-i))(z-(1-i))(z-(-1+i)) \\ &= (z-1-i)(z+1+i)(z-1+i)(z+1-i) \\ &= [(z-1-i)(z-1+i)][(z+1+i)(z+1-i)] \\ &= [z^2-2z+2][z^2+2z+2] \end{aligned}$$

(7) Assume c is an nth root of unity other than unity. Then

$$\begin{aligned} 1+c+c^2+\cdots+c^{n-1} &= \frac{(1+c+c^2+\cdots+c^{n-1})(1-c)}{1-c} \\ &= \frac{1-c^n}{1-c} \quad \text{(expand numerator, all else cancels)} \\ &= \frac{1-1}{1-c} \quad \text{(since } c \text{ is an } n\text{th root of 1)} \\ &= 0 \end{aligned}$$

Note that dividing by $1-c$ is where we use the assumption that c is a root *other than* unity.