12 - Solutions

(12.A) Show $|z| > 2$ is open using the formal neighborhood definition.

Let z_0 be a point in the set $|z| > 2$. Then $|z_0| = 2 + c$ where $c > 0$.
If z satisfies the inequality $|z - z_0| < \frac{c}{2}$, then using the backwards Triangle Inequality

$$|z| > |z_0| - |z - z_0| > 2 + c - \frac{c}{2} = 2 + \frac{c}{2} > 2$$

This means that the neighborhood radius $\frac{c}{2}$ of z_0 is contained in $|z| > 2$ (z_0 is interior). Since z_0 was arbitrary, $|z| > 2$ is open.

(12.B) Show the real line is closed.

We will show the compliment of the real line is open.
Let z_0 be in the compliment of the real line. Then $|\text{Im}\, z_0| = c > 0$. Since the distance from z_0 to the real line is c, then a neighborhood of z_0 with radius $\frac{c}{2}$ lies entirely outside the real line. Since z_0 was an arbitrary point outside the real line, the compliment of the real line is open and the real line is closed.

(12.C) Prove that the union of two open sets is open.

Let S and T be open sets. Let $z_0 \in S \cup T$. Then either $z_0 \in S$ or $z_0 \in T$. If $z_0 \in S$, then since S is open there is a nbd of z_0 contained in S. But everything in S is in $S \cup T$, so the neighborhood is contained in $S \cup T$. Similarly, if $z_0 \in T$ then since T is open, we can find a neighborhood of z_0 contained in $T \subset S \cup T$. Since z_0 was arbitrary, $S \cup T$ is open.