Solutions - 15 (more limits)

(5)

$$\begin{aligned}
\lim_{(x,0)\to(0,0)} \frac{z^2}{\bar{z}^2} &= \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} \\
&= \lim_{(x,0)\to(0,0)} 1 \\
&= 1 \\
\lim_{(a,a)\to(0,0)} \frac{z^2}{\bar{z}^2} &= \lim_{(a,a)\to(0,0)} \frac{(a+ia)^2}{(a-ia)^2} \\
&= \lim_{(a,a)\to(0,0)} \frac{2i}{-2i} \\
&= -1
\end{aligned}$$

Since the answers are different, the limit $\lim_{z\to 0} \frac{z^2}{|z|^2}$ does not exist.

(A)

$$\begin{aligned}
\lim_{z\to 0} \frac{z-\bar{z}}{|z|} &= \lim_{z\to 0} \frac{2iy}{\sqrt{x^2+y^2}} \\
\lim_{(x,0)\to(0,0)} \frac{2iy}{\sqrt{x^2+y^2}} &= \lim_{(x,0)\to(0,0)} \frac{0}{\sqrt{x^2+0}} \\
&= 0 \\
\lim_{(0,y)\to(0,0)} \frac{2iy}{\sqrt{x^2+y^2}} &= \lim_{(0,y)\to(0,0)} \frac{2iy}{\sqrt{0+y^2}} \\
&= \pm 2i
\end{aligned}$$

The "$\pm$" is there since from the numerator has y but the denominator has $|y|$. So actually, this second limit (along the imaginary axis) doesn't exist since it has two different answers. And both are different from the real axis answer. Since all these answers are different, the limit $\lim_{z\to 0} \frac{z-\bar{z}}{|z|}$ does not exist.

(9) Let $\epsilon > 0$. Assume $\lim_{z\to z_0} f(z) = 0$ and $|g(z)| \leq M$ for all z in the neighborhood $|z - z_0| < \delta_1$. Assume that z is in the neighborhood $|z - z_0| < \delta_1$.

$$\begin{aligned}
|f(z)g(z) - 0| &= |f(z)g(z)| \\
&= |f(z)||g(z)| \\
&\leq |f(z)|M
\end{aligned}$$

Since $\lim_{z\to z_0} f(z) = 0$, there exists a $\delta_2 > 0$ such that when $|z - z_0| < \delta_2$, then $|f(z) - 0| = |f(z)| < \frac{\epsilon}{M}$. Then if z is in the neighborhood $|z - z_0| < \delta = \min\{\delta_1, \delta_2\}$,

$$\begin{aligned}
|f(z)g(z)| &\leq |f(z)|M \\
&< \frac{\epsilon}{M} M \\
&= \epsilon
\end{aligned}$$