Section 15 limit practice

Prove the following limits using ϵ's and δ's.

(1)

$$\lim_{z \to 1+i} (3iz - 4) = 3i - 7$$

Let $\epsilon > 0$ be given. Assume $|z - (1 + i)| < \delta$.

$$\begin{aligned} |3iz - 4 - (3i - 7)| &= |3iz - 3i + 3| \\ &= |3i(z - 1 - i)| \\ &= 3|z - (1 + i)| \\ &< 3\delta \end{aligned}$$

If $|z - (1 + i)| < \delta = \frac{\epsilon}{3}$, then $|3iz - 4 - (3i - 7)| < \epsilon$.

(2)

$$\lim_{z \to 0} \frac{z^2}{\bar{z}} = 0$$

Let $\epsilon > 0$ be given. Assume $|z - 0| = |z| < \delta$.

$$\begin{aligned} \left| \frac{z^2}{\bar{z}} - 0 \right| &= \frac{|z^2|}{|\bar{z}|} \\ &= \frac{|z|^2}{|z|} \\ &= |z| \\ &< \delta \end{aligned}$$

If $|z - 0| < \delta = \epsilon$, then $\left| \frac{z^2}{\bar{z}} - 0 \right| < \epsilon$.

(3)

$$\lim_{z \to i} z^2 + 4 = 3$$

Let $\epsilon > 0$ be given. Assume $|z - i| < \delta$.

$$\begin{aligned} |z^2 + 4 - 3| &= |z^2 + 1| \\ &= |(z - i)(z + i)| \\ &= |z - i||z + i| \\ &< \delta|z + i| \qquad \text{by picture, distance } < 2 + \delta \\ &< \delta(\delta + 2) \qquad \text{1st condition: } \delta \leq 1 \\ &\leq \delta(3) \qquad \text{2nd condition: } \delta \leq \frac{\epsilon}{3} \\ &\leq \epsilon \end{aligned}$$

If $|z-i| < \delta = \min\{1, \frac{\epsilon}{3}\}$, then $|z^2+4-3| < \epsilon$.

(4)

$$\lim_{z\to 3} \frac{1}{z^2} = \frac{1}{9}$$

Let $\epsilon > 0$ be given. Assume $|z-3| < \delta$.

$$\begin{aligned} |\frac{1}{z^2} - \frac{1}{9}| &= |\frac{9-z^2}{9z^2}| \\ &= |\frac{(3-z)(3+z)}{9z^2}| \\ &= \frac{|z-3||z+3|}{9|z|^2} \end{aligned}$$

If we impose a first condition of $\delta \leq 1$, then $|z| = |3+z-3| \geq |3-|z-3|| \geq |3-1| = 2$. We also have that $|z+3| \leq |z|+3 \leq 4+3 \leq 7$. Therefore,

$$\begin{aligned} |\frac{1}{z^2} - \frac{1}{9}| &= \frac{|z-3||z+3|}{9|z|^2} \\ &\leq \frac{|z-3|7}{9(2)^2} \\ &< \frac{\delta 7}{36} \end{aligned}$$

If $|z-3| < \delta = \min\{1, \frac{36\epsilon}{7}\}$, then $|\frac{1}{z^2} - \frac{1}{9}| < \epsilon$.