Section 20 Solutions

(20.8a) Show that \( f(z) = \text{Re} z \) is not differentiable for any \( z \) by showing the limit in the definition of the derivative doesn’t exist.

\[
\begin{align*}
f'(z) &= \lim_{\Delta z \to 0} \frac{\text{Re}(z + \Delta z) - \text{Re} z}{\Delta z} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{x + \Delta x - x}{\Delta z + i\Delta y} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x}{\Delta z + i\Delta y}
\end{align*}
\]

If we let \( \Delta z \) go to 0 along the line \((\Delta x,0)\), the limit is 1. Along the line \((0,\Delta y)\), the limit is 0. Since the answers are different, the limit does not exist and \( f \) is not differentiable.

(23.1b) Using the Cauchy-Riemann equations, show that \( f(z) = z - \bar{z} \) is not differentiable for any \( z \).

\[
\begin{align*}
f(z) &= z - \bar{z} = x + iy - (x - iy) = 2iy, \text{ so } u(x, y) = 0 \text{ and } v(x, y) = 2y. \\
u_x &= 0 \\
u_y &= 0 \\
v_x &= 0 \\
v_y &= 2
\end{align*}
\]

\( v_x = -u_y = 0 \) but \( u_x \neq v_y \) ever. \( f'(z) \) does not exist.

(23.1d) Using the Cauchy-Riemann equations, show that \( f(x + iy) = e^x e^{-iy} \) is not differentiable for any \( z \).

\[
\begin{align*}
f(z) &= e^x e^{-iy} = e^x \cos(-y) + i e^x \sin(-y) = e^x \cos y - i e^x \sin y, \text{ so } u(x, y) = e^x \cos y \text{ and } v(x, y) = -e^x \sin y. \\
u_x &= e^x \cos y \\
u_y &= -e^x \sin y \\
v_x &= -e^x \sin y \\
v_y &= -e^x \cos y
\end{align*}
\]

\( v_x \neq -u_y \) unless \( \sin y = 0 \) and \( u_x \neq v_y \) unless \( \cos y = 0 \). Since \( \sin y \) and \( \cos y \) cannot both be 0 at the same time, \( f'(z) \) does not ever exist.

(23.3a) Suppose \( f(z) = \frac{1}{z} \). Using the Cauchy-Riemann equations, determine where \( f'(z) \) exists and give its value for the \( z \) when it does exist.

\[
\begin{align*}
f(z) &= \frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}, \text{ so } u(x, y) = -\frac{x}{x^2 + y^2} \text{ and } v(x, y) = -\frac{y}{x^2 + y^2}.
\end{align*}
\]

\[
\begin{align*}
u_x &= \frac{(1)(x^2 + y^2) - (x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
u_y &= \frac{(0)(x^2 + y^2) - (x)(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}
\end{align*}
\]
\[ v_x = \frac{(0)(x^2 + y^2) - (-y)(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2} \]

\[ v_y = \frac{(-1)(x^2 + y^2) - (-y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \]

The CR equations are satisfied and the partials are continuous if \( z \neq 0 \), so \( f'(z) = u_x + iv_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \frac{2xy}{(x^2 + y^2)^2} = \frac{(x - iy)(x - iy)}{(z \bar{z})^2} = \frac{x^2}{z \bar{z}^2} = \frac{1}{z^2} \).

(23.3b) Suppose \( f(x + iy) = x^2 + iy^2 \). Using the Cauchy-Riemann equations, determine where \( f'(z) \) exists and give its value for the \( z \) when it does exist.

\[ f(z) = x^2 + iy^2 \] so \( u(x, y) = x^2 \) and \( v(x, y) = y^2 \).

\[
\begin{align*}
  u_x &= 2x \\
  u_y &= 0 \\
  v_x &= 0 \\
  v_y &= 2y
\end{align*}
\]

The CR equations are only satisfied if \( x = y \). When \( x = y \), \( f'(z) = 2x \).