Section 20 Solutions

(20.8a) Show that $f(z) = \text{Re}\, z$ is not differentiable for any z by showing the limit in the definition of the derivative doesn't exist.

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{\text{Re}\,(z + \Delta z) - \text{Re}\, z}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{x + \Delta x - x}{\Delta x + i\Delta y} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x}{\Delta x + i\Delta y} \end{aligned}$$

If we let Δz go to 0 along the line $(\Delta x, 0)$, the limit is 1. Along the line $(0, \Delta y)$, the limit is 0. Since the answers are different, the limit does not exist and f is not differentiable.

(23.1b) Using the Cauchy-Riemann equations, show that $f(z) = z - \bar{z}$ is not differentiable for any z.

$f(z) = z - \bar{z} = x + iy - (x - iy) = 2iy$, so $u(x,y) = 0$ and $v(x,y) = 2y$.

$$\begin{aligned} u_x &= 0 \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 2 \end{aligned}$$

$v_x = -u_y = 0$ but $u_x \neq v_y$ ever. $f'(z)$ does not exist.

(23.1d) Using the Cauchy-Riemann equations, show that $f(x + iy) = e^x e^{-iy}$ is not differentiable for any z.

$f(z) = e^x e^{-iy} = e^x \cos(-y) + ie^x \sin(-y) = e^x \cos y - ie^x \sin y$, so $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$.

$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= -e^x \sin y \\ v_y &= -e^x \cos y \end{aligned}$$

$v_x \neq -u_y$ unless $\sin y = 0$ and $u_x \neq v_y$ unless $\cos y = 0$. Since $\sin y$ and $\cos y$ cannot both be 0 at the same time, $f'(z)$ does not ever exist.

(23.3a) Suppose $f(z) = \frac{1}{z}$. Using the Cauchy-Riemann equations, determine where $f'(z)$ exists and give its value for the z when it does exist.

$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, so $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = -\frac{y}{x^2+y^2}$.

$$\begin{aligned} u_x &= \frac{(1)(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \\ u_y &= \frac{(0)(x^2+y^2) - (x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \end{aligned}$$

$$
\begin{aligned}
v_x &= \frac{(0)(x^2+y^2)-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \\
v_y &= \frac{(-1)(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}
$$

The CR equations are satisfied and the partials are continuous if $z \neq 0$, so $f'(z) = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = \frac{-(x-iy)(x-iy)}{(z\bar{z})^2} = \frac{-\bar{z}^2}{z^2\bar{z}^2} = \frac{-1}{z^2}$.

(23.3b) Suppose $f(x+iy) = x^2 + iy^2$. Using the Cauchy-Riemann equations, determine where $f'(z)$ exists and give its value for the z when it does exist.

$f(z) = x^2 + iy^2$ so $u(x,y) = x^2$ and $v(x,y) = y^2$.

$$
\begin{aligned}
u_x &= 2x \\
u_y &= 0 \\
v_x &= 0 \\
v_y &= 2y
\end{aligned}
$$

The CR equations are only satisfied if $x = y$. When $x = y$, $f'(z) = 2x$.