Section 23 Solutions

(8) We know that if f is differentiable at z_0 then

$$f'(z_0) = u_x + iv_x$$

Our goal is to rewrite $f'(z_0)$ in terms of u_r and v_r. Using the multidimensional chain rule, $u_r = u_x x_r + u_y y_r$. Since $x = r\cos\theta$ and $y = r\sin\theta$, $u_r = u_x\cos\theta + u_y\sin\theta$. A similar equation holds for v_r.

$$\begin{array}{rcl} u_r = u_x\cos\theta + u_y\sin\theta & & v_r = v_x\cos\theta + v_y\sin\theta \\ u_r = u_x\cos\theta + u_y\sin\theta & & v_r = -u_y\cos\theta + u_x\sin\theta \quad \text{(using the CR eqs)} \\ u_r\cos\theta = u_x\cos^2\theta + u_y\sin\theta\cos\theta & & v_r\sin\theta = -u_y\sin\theta\cos\theta + u_x\sin^2\theta \quad \text{(now add these eqs)} \\ u_r\cos\theta + v_r\sin\theta & = & u_x \end{array}$$

Now solve for u_y in a similar way.

$$\begin{array}{rcl} u_r = u_x\cos\theta + u_y\sin\theta & & v_r = -u_y\cos\theta + u_x\sin\theta \\ u_r\sin\theta = u_x\sin\theta\cos\theta + u_y\sin^2\theta & & v_r\cos\theta = -u_y\cos^2\theta + u_x\sin\theta\cos\theta \quad \text{(now subtract)} \\ u_r\sin\theta - v_r\cos\theta & = & u_y \\ f'(z_0) & = & u_x + iv_x \\ & = & u_x - iu_y \\ & = & u_r\cos\theta + v_r\sin\theta - iu_r\sin\theta + iv_r\cos\theta \\ & = & u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta) \\ & = & u_r(\cos\theta - i\sin\theta) + iv_r(-i\sin\theta + \cos\theta) \\ & = & u_r e^{-i\theta} + iv_r e^{-i\theta} \\ & = & e^{-i\theta}(u_r + iv_r) \end{array}$$