Solutions - Sections 30 and 31

(2c)

$$\begin{aligned} -1+i\sqrt{3} &= 2e^{i\frac{2\pi}{3}} \\ \log(-1+i\sqrt{3}) &= \ln 2 + i(\frac{2\pi}{3}+2\pi n) \\ &= \ln 2 + i2\pi(\frac{1}{3}+n) \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

(3a)

$$\begin{aligned} 1+i &= \sqrt{2}e^{i\frac{\pi}{4}} \\ (1+i)^2 &= 2e^{i\frac{\pi}{2}} \\ \log(1+i)^2 &= \ln 2 + i(\frac{\pi}{2}+2\pi n) \qquad (n = 0, \pm 1, \pm 2, \ldots) \\ \text{Log}\,(1+i)^2 &= \ln 2 + i(\frac{\pi}{2}) \\ \log(1+i) &= \ln\sqrt{2} + i(\frac{\pi}{4}+2\pi n) \\ &= \frac{1}{2}\ln 2 + i(\frac{\pi}{4}+2\pi n) \\ \text{Log}\,(1+i) &= \frac{1}{2}\ln 2 + i(\frac{\pi}{4}) \\ 2\text{Log}\,(1+i) &= \ln 2 + i(\frac{\pi}{2}) = \text{Log}\,(1+i)^2 \end{aligned}$$

(3b)

$$\begin{aligned} -1+i &= \sqrt{2}e^{i\frac{3\pi}{4}} \\ (-1+i)^2 &= 2e^{i\frac{3\pi}{2}} \\ \log(-1+i)^2 &= \ln 2 + i(\frac{3\pi}{2}+2\pi n) \qquad (n = 0, \pm 1, \pm 2, \ldots) \\ \text{Log}\,(-1+i)^2 &= \ln 2 + i(-\frac{\pi}{2}) \\ \log(-1+i) &= \ln\sqrt{2} + i(\frac{3\pi}{4}+2\pi n) \\ &= \frac{1}{2}\ln 2 + i(\frac{3\pi}{4}+2\pi n) \\ \text{Log}\,(-1+i) &= \frac{1}{2}\ln 2 + i(\frac{3\pi}{4}) \\ 2\text{Log}\,(-1+i) &= \ln 2 + i(\frac{3\pi}{2}) \neq \text{Log}\,(-1+i)^2 \end{aligned}$$

(10) First, check that $u = \ln(x^2+y^2)$ satisfies Laplace's equation.

$$u_x = \frac{2x}{x^2+y^2}$$

$$\begin{aligned}
u_{xx} &= \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} \\
&= \frac{2y^2 - 2x^2}{(x^2+y^2)^2} \\
u_y &= \frac{2y}{x^2+y^2} \\
u_{yy} &= \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} \\
&= \frac{2x^2 - 2y^2}{(x^2+y^2)^2} \\
u_{xx} + u_{yy} &= 0
\end{aligned}$$

Since u satisfies Laplace's equation and its second partials are continuous when $z \neq 0$, u is harmonic when $z \neq 0$.

Now, note that $u = \ln(x^2 + y^2) = \ln(|z|^2) = \ln(|z^2|)$ is the real part of the logarithm $f(z) = \log(z^2) = \ln(|z^2|) + i(\arg z^2)$. The branches of $\log(z^2)$ are analytic since they are compositions of analytic functions. Since u is the real part of an analytic function, u is harmonic off the branch cut. But since u is the same no matter which branch cut we use, u is harmonic everywhere except $z = 0$ (which is on every branch cut).