Solutions - Section 34

(A) Verify the identity $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

$$\begin{aligned}
\sin z_1 \cos z_2 + \cos z_1 \sin z_2 &= \frac{e^{iz_1} - e^{-iz_1}}{2i}\frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2}\frac{e^{iz_2} - e^{-iz_2}}{2i} \\
&= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)}}{4i} \\
&\quad + \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)}}{4i} \\
&= \frac{2e^{i(z_1+z_2)} - 2e^{i(-z_1-z_2)}}{4i} \\
&= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} \\
&= \sin(z_1 + z_2)
\end{aligned}$$

(12) If $z = x$ is real, then both $\sin x$ and $\cos x$ are equal to their real counterparts. This is how they were defined. Since both $\sin z$ and $\cos z$ are entire and real on the real axis, by the Reflection Principle $\overline{\sin z} = \sin \bar{z}$ and $\overline{\cos z} = \cos \bar{z}$.

(14a)

$$\begin{aligned}
\overline{\cos(iz)} &= \overline{\left(\frac{e^{i(iz)} + e^{-i(iz)}}{2}\right)} \\
&= \overline{\left(\frac{e^{-z} + e^{z}}{2}\right)} \\
&= \frac{e^{-\bar{z}} + e^{\bar{z}}}{2} \\
&= \frac{e^{i(i\bar{z})} + e^{-i(i\bar{z})}}{2} \\
&= \cos(i\bar{z})
\end{aligned}$$

(14b)

$$\begin{aligned}
\overline{\sin(iz)} &= \overline{\left(\frac{e^{i(iz)} - e^{-i(iz)}}{2i}\right)} \\
&= \overline{\left(\frac{e^{-z} - e^{z}}{2i}\right)} \\
&= \frac{e^{-\bar{z}} - e^{\bar{z}}}{-2i} \\
&= -\frac{e^{i(i\bar{z})} - e^{-i(i\bar{z})}}{2i} \\
&= -\sin(i\bar{z})
\end{aligned}$$

Under what conditions does $\sin(i\bar{z}) = -\sin(i\bar{z})$? Let's set them equal and check.

$$\begin{aligned}
\sin(i\bar{z}) &= -\sin(i\bar{z}) \\
\frac{e^{-\bar{z}} - e^{\bar{z}}}{2i} &= -\frac{e^{-\bar{z}} - e^{\bar{z}}}{2i} \\
e^{-\bar{z}} - e^{\bar{z}} &= -e^{-\bar{z}} + e^{\bar{z}} \\
1 - e^{2\bar{z}} &= -1 + e^{2\bar{z}} \\
e^{2\bar{z}} &= 1 \\
\bar{z} &= i\pi n \qquad n = 0, \pm 1, \pm 2, \ldots \\
z &= -i\pi n = i\pi n \qquad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$