Solutions - Section 38

(2a)

$$\begin{aligned}\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \frac{1}{t^2} - 1\, dt + i\int_1^2 -\frac{2}{t}\, dt \\ &= (\frac{-1}{t} - t|_1^2 + i(-2\ln t|_1^2 \\ &= (-\frac{1}{2} - 2 + 1 + 1) + i(-2\ln 2 + 0) \\ &= -\frac{1}{2} - i2\ln 2\end{aligned}$$

(2b)

$$\begin{aligned}\int_0^{\frac{\pi}{6}} e^{i2t} dt &= \frac{e^{i2t}}{i2}|_0^{\frac{\pi}{6}} \\ &= \frac{e^{i\frac{\pi}{3}}}{2i} - \frac{1}{2i} \\ &= \frac{1 + i\sqrt{3}}{4i} + \frac{i}{2} \\ &= -\frac{i}{4} + \frac{\sqrt{3}}{4} + \frac{i}{2} \\ &= \frac{\sqrt{3}}{4} + \frac{i}{4}\end{aligned}$$

(2c)

$$\begin{aligned}\int_0^{\infty} e^{-zt} dt &= \frac{e^{-zt}}{-z}|_0^{\infty} \\ &= \frac{e^{-xt}e^{-iyt}}{-z}|_0^{\infty} \\ &= 0 + \frac{1}{z}\end{aligned}$$

Note that $\lim_{t\to\infty} e^{-xt} = 0$ since $x = \operatorname{Re} z > 0$. $|\frac{e^{iyt}}{-z}| = \frac{1}{z}$, so $\lim_{t\to\infty} \frac{e^{-xt}e^{-iyt}}{-z} = 0$ by the Squeeze Theorem.

(3) First assume m and n are integers and $m \neq n$.

$$\begin{aligned}\int_0^{2\pi} e^{im\theta}e^{-in\theta}\, d\theta &= \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta \\ &= (\frac{e^{i(m-n)\theta}}{i(m-n)}|_0^{2\pi} \\ &= \frac{e^{i(m-n)2\pi}}{i(m-n)} - \frac{1}{i(m-n)} \\ &= \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0\end{aligned}$$

Now assume $m = n$.

$$\begin{aligned}\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta &= \int_0^{2\pi} 1\, d\theta \\ &= 2\pi\end{aligned}$$