MATH 430 COMPLEX ANALYSIS

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These are notes from an introduction to complex analysis at the undergraduate level as taught by Paul Taylor at Shippensburg University during the Fall 2016 semester. If you notice any errors of any kind (as I'm sure there are many) you can email me at tp7924@ship.edu.

Contents

1. Lecture 1: Arithmetic in Complex numbers

What was the motivation behind developing complex analysis?

- They are needed to solve polynomial equations.
  - ⋄ For example the equation $x^2 + x + 1 = 0$ can't be solved without complex numbers.
  - ⋄ In reality most of the motivation came from solutions to cubic equations. (NOT quadratics!)

We can split complex numbers into real and imaginary parts. $i = \sqrt{-1}$ is the unit 'vector' in the imaginary direction.

In general we have that if $z = x + iy$ then $Re(z) = x$ and $Im(z) = y$.

Example 1.1. *If* $z = 3 + 5i$ *then* $Re(z) = 3$ *and* $Im(z) = 5$.

We have all the standard properties we are accustom to; Commutativity ($a+b = b+a$ & $ab = ba$), Associative ($(a+b)+c = a+(b+c)$ & $(ab)c = c(ab)$), and Distributive ($a(b+c) = ab + ac$)).

Adding and subtracting in $\mathbb{C}$ is as expected. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

We also have a fairly logical definition of multiplication:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1).$$

Example 1.2. ***Solve*** $z^2 + z + 1 = 0$

Let $z = x + iy$ *where* $x, y \in \mathbb{R}$. *Then we have*

$$z^2+z+1 = (x+iy)^2+(x+iy)+1 = (x^2-y^2+x+1)+i(2xy+y) = 0+0i.$$

So, this means that $x^2 - y^2 + x + 1 = 0$ *and* $2xy + y = 0$. *Considering* $2xy + y = 0$ *we see that it has solutions* $(x, 0)$ *and* $(-1/2, 0)$. *So we can break this into these two cases:*

Case 1: *When we have a solution to* $2xy + y = 0$ *of the form* $(x, 0)$ *then our other equation* ($x^2 - y^2 + x + 1 = 0$) *becomes* $x^2 + x + 1 = 0$. *But we have that the roots of this equation are* $x = \frac{-1}{2} \pm \frac{\sqrt{-3}}{2}$, *contradiction the fact that* x *is real. Thus there are no solutions when* $y = 0$.

Case 2: *When we have a solution to* $2xy + y = 0$ *of the form* $(-1/2, y)$ *then our equation* ($x^2 - y^2 + x + 1 = 0$) *becomes* $3/4 - y^2 = 0$. *So solving for* y *we get* $y = \pm\sqrt{3}/2$. *This gives solutions of* $(-1/2, \pm\sqrt{3}/2)$ *to our original equation. Explicitly these solutions are:*

$$z = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Theorem 1.3. *If* $z_1 z_2 = 0$ *then at least one of* z_1 *or* z_2 *is zero.*

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Now WLOG suppose $z_2 \neq 0$. Thus we can divide by z_2:

$$\frac{z_1 z_2}{z_2} = \frac{0}{z_1}.$$

And by our definition of division we have that $z_1 = 0$ □

Elements of $\mathbb{C}$ can be thot of as two dimensional vectors with real and imaginary components.

Example 1.4. *The vectors corresponding to* $1 - 2i$ *and* $3 + 4i$*:*

Definition 1.5. The magnitude (aka modulus or length) of $z = x + iy$, denoted $|z|$ is:

$$|z| := \sqrt{x^2 + y^2}$$

We also have a notion of distance between z_1 and z_2.

Definition 1.6. The distance between z_1 and z_2 is defined as:

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

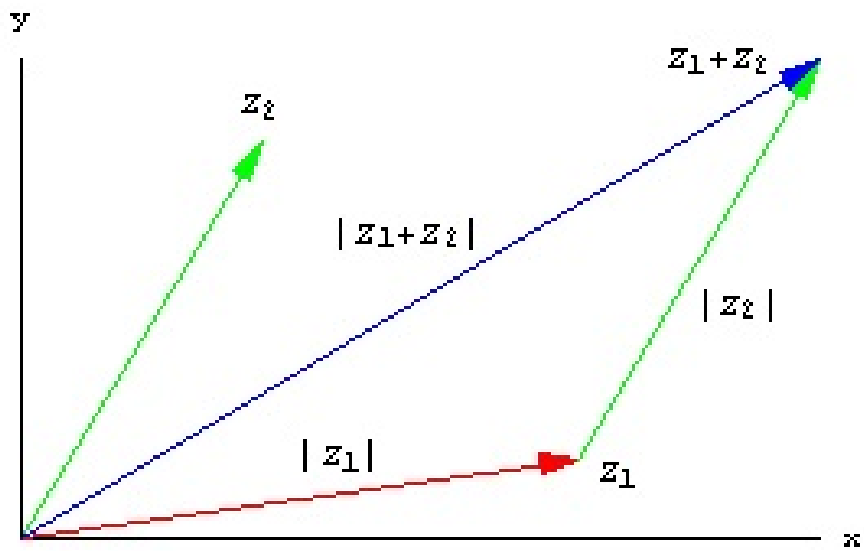
We have a nice equation for a circle in $\mathbb{C}$ centered at z_0 with radius r:

$$|z - z_0| = r.$$

The **triangle inequality** is

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

It can be easily derived geometrically:



We can also rewrite the triangle inequality to obtain the backwards triangle inequality:

$$|z_1 - z_2| \geq ||z_1| - |z_2||.$$

2. Lecture 2

We start off with an example problem:

Example 2.1. *Assume z satisfies $|z-5|<1$. Find a bound for $|z|$.*

We can answer this with the triangle inequality. By the reverse triangle inequality

$$|z| = |(z-5)+5| \leq |z-5| + |5|$$

And by the assumption $|z-5|<1$ we obtain an upper bound for $|z|$

$$z < 1+5 = 6.$$

For the lower bound we do a similar procedure using the reverse triangle inequality,

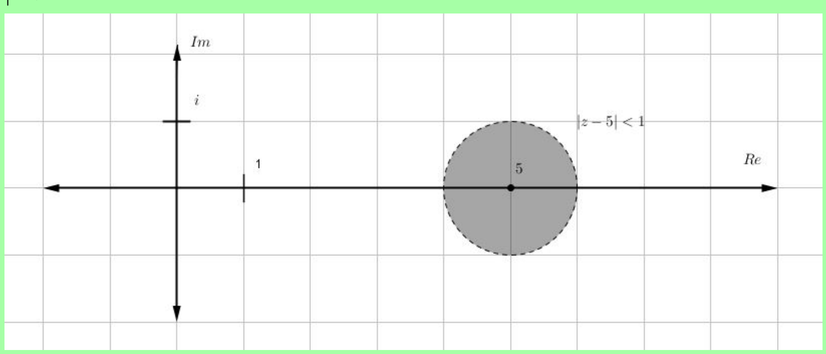
$$|z| = |(z-5)-(-5)| \geq ||z-5| - |-5||.$$

Now by our assumption that $|z-5|<1$ we obtain the lower bound

$$|z| < |1-5| = 4.$$

Altogether we have shown that $4<|z|<6$.

We can also come to this conclusion by considering the the graph $|z-5|<1$:

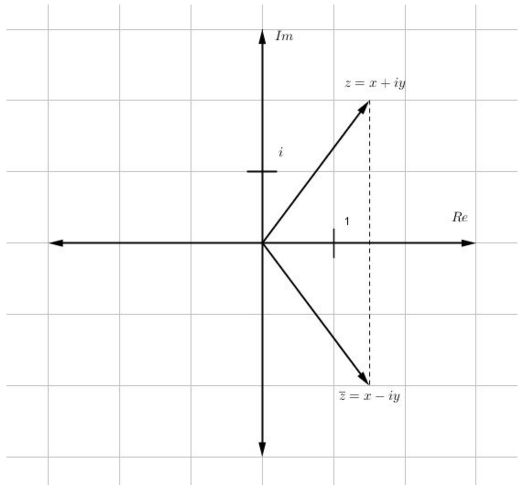


This diagram makes the fact that $4<|z|<6$ obvious.

We now give the definition of a complex conjugate.

Definition 2.2. We write the **complex conjugate** of $z = x+iy$ as $\overline{z} = x - iy$. In essence the complex conjugate flips the sign of the imaginary part of the complex number.

The complex conjugate can be thought of graphically as a reflection over the real axis.



Some properties of conjugates:

$$z\overline{z} = |z|^2$$
$$\overline{z_1 z_2} = \overline{z} \cdot \overline{z}$$
$$z + \overline{z} = 2\text{Re}(z)$$
$$z - \overline{z} = 2\text{Im}(z)$$

These are all straightforward to show using the definition of a complex conjugate.

We have the following theorem which can be proven using complex conjugates:

Theorem 2.3.
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Proof.
$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{z_1}{z_2}\right)\left(\frac{\overline{z_1}}{\overline{z_2}}\right) = \frac{z_1\overline{z_1}}{z_2\overline{z_2}} = \frac{|z_1|^2}{|z_2|^2}$$

The result now follows by squaring each side. □

This theorem can be useful in many cases:

Example 2.4. *If $|z| = 2$, then show $\left|\frac{1}{z^2+6z+9}\right| \leq 1$.*

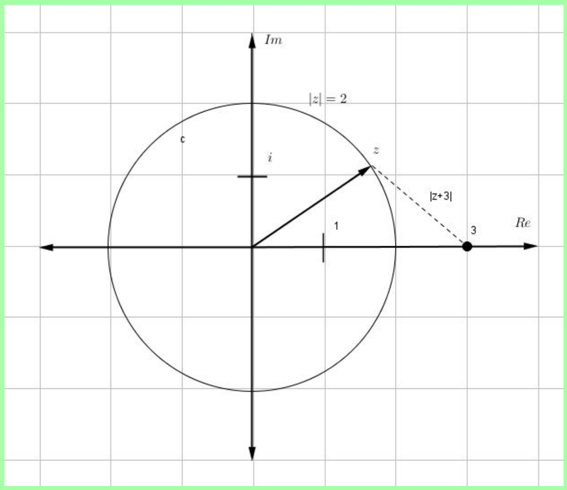
To show this we first apply the above theorem:
$$\left|\frac{1}{z^2 + 6z + 9}\right| = \frac{|1|}{|z^2 + 6z + 9|}.$$

Notice that $z^2 + 6z + 9$ can be factored to $(z+3)^2$. So in fact we are trying to show that
$$\frac{1}{|z + 3|^2} \leq 1$$

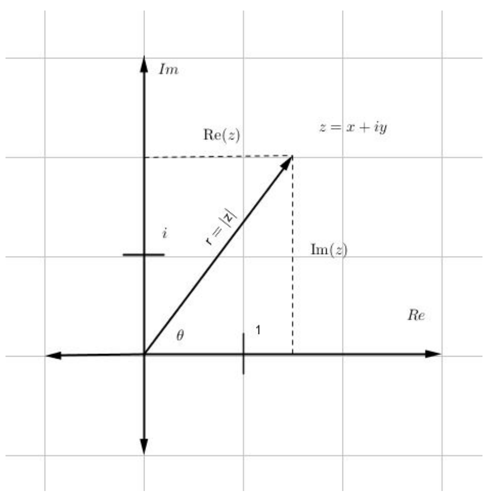
which is equivalent to showing that $|z+3| \geq 1$. This follows from the reverse triangle inequality

$$|z-(-3)| \geq ||z|-|-3|| = |2-3| = 1.$$

This problem can also be worked out with a more geometric approach. Since $|z| = 2$ we know z is some point lying on the circle of radius 2 centered at the origin.Following the algebraic argument given above, if we can show that $|z+3| \geq 1$ we will be done. Notice that $|z+3|$ can be interpreted as the distance between z and 3. So it is obvious from the following diagram that $|z+3| \geq 1$:



Since we know we can think of complex numbers as vectors it is logical to be able to use polar coordinates.



So, we have the following for $z \in \mathbb{C}$

$$\begin{aligned} z &= r\cos(\theta) + ir\sin(\theta) \\ &= r(\cos(\theta) + i\sin(\theta)) \\ &= re^{i\theta}. \end{aligned}$$

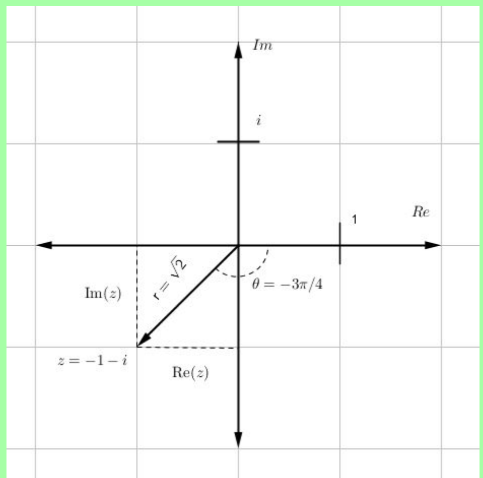
Of course $r = |z|$ is the radius from the origin and $\theta = \tan(\mathrm{Im}(z)/\mathrm{Re}(z)) + 2n\pi$ is the angle from 1, also known as the argument. The **principal argument** is the argument lying in the interval $(-\pi, \pi]$ and is denoted $\mathrm{Arg}(z)$.

Example 2.5. *Find the radius and argument for $z = -1 - i$.*

The radius is

$$r = |-1 - i| = \sqrt{((-1)^2 + (-1)^2)} = \sqrt{2}.$$

For the argument we use the graph:



to find that $\mathrm{Arg}(z) = -3\pi/4$.

We note the following celebrated formula found by Euler (Euler's formula):

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This will be especially useful for us in complex analysis since it gives a way of switching to and from exponential form.

A notable instance of the above formula is when we let $\theta = \pi$:

$$e^{i\pi} = -1 \Rightarrow e^{i\theta} + 1 = 0.$$

An example of an application of Euler's formula is putting $1 + i\sqrt{3}$ in to exponential form:

$$1 + i\sqrt{3} = 2(\cos(\pi/3) + i\sin(\pi/3)) = 2e^{\pi/3}.$$

A good way to think of the process of switching into exponential form is graphically as we saw earlier.

All the same properties that we are used to for e^n still work for the complex numbers.

Using Euler's formula we can come up with a number of tricky trig identities.

Example 2.6. *We can come up with the angle addition identities.*

We will be doing this by looking at the product $e^{i\theta}$ in two ways;

$$e^{ia}e^{ib} = e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

and

$$\begin{aligned} e^{ia}e^{ib} &= (\cos(a) + i\sin(a)) \cdot (\cos(b) + i\sin(b)) \\ &= (\cos(a)\cos(b) - \sin(a)\sin(b)) + i(\sin(a)\cos(b) + \cos(a)\sin(b)). \end{aligned}$$

If we set the real and imaginary parts of these two equivalent expressions we get the angle addition formulas from trigonometry:

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

and

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b).$$

Similarly we can derive the double angle formulas starting from $(e^{i\theta})^2 = e^{i2\theta}$. In fact we have **De Moivere's formula**:

Theorem 2.7.

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

This formula can be easily derived using Euler's formula. It is especially usefull in find double-angle, triple-angle, ..., and arbitrary-angle formulas for trigonometry.

3. Lecture 3: Roots & Regions

Roots

In exponential form the argument can take on multiple values. This begs the following question:

When are 2 complex numbers equal in exponential form?

Say we have two complex numbers which are equal in exponential form:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2}.$$

For these to be equal we must have $r_1 = r_2$ and $\theta_1 \equiv \theta_2 \mod 2\pi$. For example $4e^{i2\pi/3}$ is equal to many (infinitely many) other numbers in exponential form with different arguments:

$$4e^{i2\pi/3} = 4e^{i8\pi/3} = 4e^{-i4\pi/3} = \cdots = 4e^{i(2\pi/3+2\pi n)}$$

Lets get right into taking roots:

Example 3.1. *Find the square root(s) of $4e^{i2\pi/3}$.*

Notice that this is equivalent to solving

$$z^2 = 4e^{i2\pi/3}$$

for z. Intuitively from our notion of complex multiplication we would expect the square root to take half the angle and the regular square root of the magnitude. This intuition is accurate.

We have

$$(4e^{i2\pi/3})^{1/2} = 2e^{i\pi/3}.$$

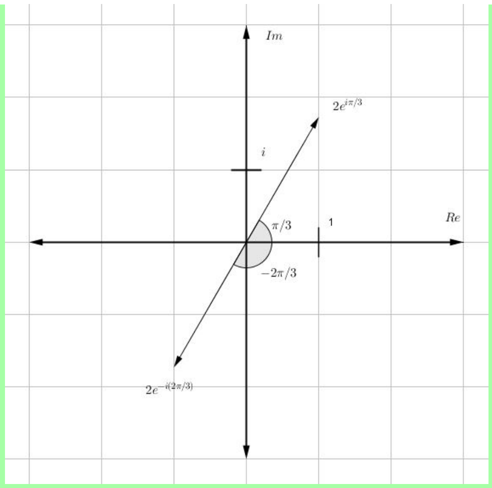
Great, so this gives $2e^{i\pi/3}$ as a square root of $4e^{i2\pi/3}$; but is this the only one? Remember that complex numbers can take on many arguments. Lets see what happens when we consider $4e^{-i4\pi/3}$ (which is equivalent to $4e^{i2\pi/3}$). Now taking the square root we have

$$(4e^{i-4\pi/3})^{1/2} = 2e^{i(-2\pi/3)}.$$

So, is $2e^{-i2\pi/3}$ a valid square root of $4e^{i2\pi/3}$; certainly $2e^{-i2\pi/3} \neq 2e^{i\pi/3}$. The sort answer is yes, they are both square roots of $4e^{i2\pi/3}$. In fact we can take the more general square root

$$(4e^{i(2\pi/3+2\pi k})^{1/2} = 2e^{i(\pi/3+\pi k)}.$$

From this we notice that depending on the parity of k we will get two different square roots π radians from one another. These two square roots are illustrated in the following diagram:



One way to get rid of our root ambiguity is to use the principal argument (before taking the root); we call the result the principal root. This root is always the root nearest to 0.

An nth root of unity is an nth root of 1. For example:

Example 3.2. *Find all cube roots of unity. (i.e. all solutions to $z^3 = 1$).*

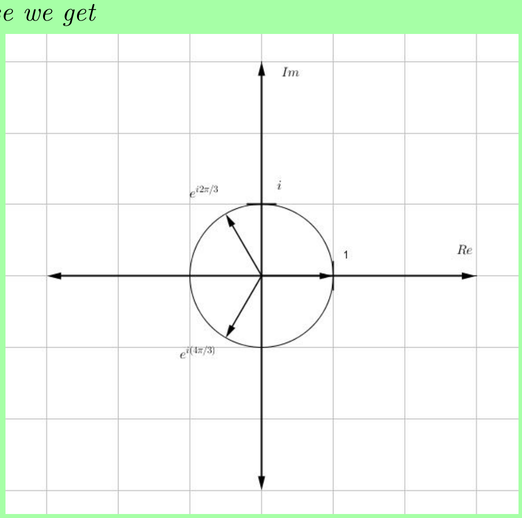
We rewrite 1 as $1 = e^{i2\pi n}$ for $n \in \mathbb{Z}$. Now we take cube roots to solve $z^3 = e^{i2\pi n}$:

$$(z^3)^{1/3} = (e^{i2\pi n})^{1/3}$$
$$z = e^{\frac{i2\pi n}{3}}.$$

Now depending on our choice of n what we will get will be equivalent to one of the following three cube roots of unity

$$1, e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}}.$$

Graphing these we get



Regions:

In $\mathbb{R}$, we use intervals a lot.

- Ex. $(2, 3) \Rightarrow x \in (2 < x < 5)$

Since $\mathbb{C}$ is two dimensional, intervals don't work so well.

In analysis, intervals are useful as **neighborhoods** (numbers close to a center number).

Example 3.3. *Find the numbers within* 0.1 *of 2.*

In $\mathbb{R}$ *this is* $(1.9, 2.1)$.

In $\mathbb{C}$ *this is* $|z - 2| < 0.1$.

- *A neighborhood (NBD) of 2 with* $r = 0.1$.

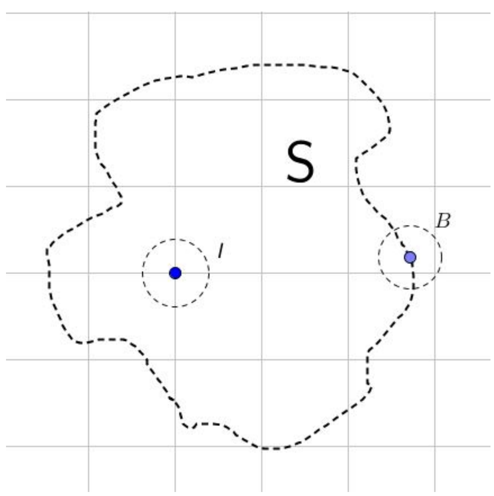
As default we use ϵ (epsilon) to represent a small radius. Epsilon in *real, small,* and *positive*.

Any two-dimensional shape can be thot of as a set of complex numbers.

Definition 3.4. The edge of S is called the **boundary**. More precisely, a point z_0 is a boundary point of S if every neighborhood of z_0 contains some points in S and some points not in S. We will denote the boundary of S as ∂S.

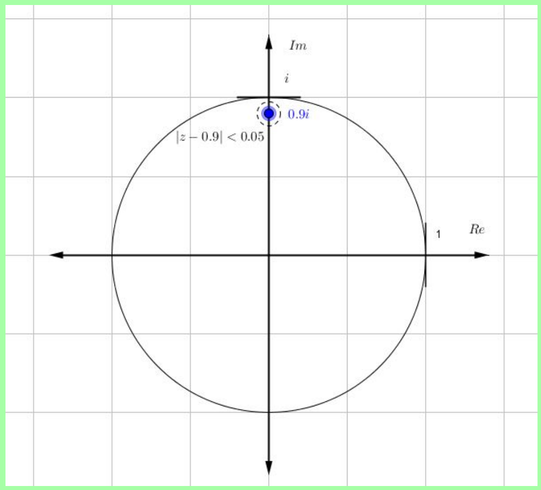
Definition 3.5. The inside part of S is called the **interior**. More precisely, a point is an interior point of S if you can find a neighborhood of the point which is completely contained in S.

In this diagram I is an interior point of the set S and B is a boundary point of S.



Example 3.6. *Let* $S = z : |z| < 1$*. Prove that* $0.9i$ *is an interior point.*

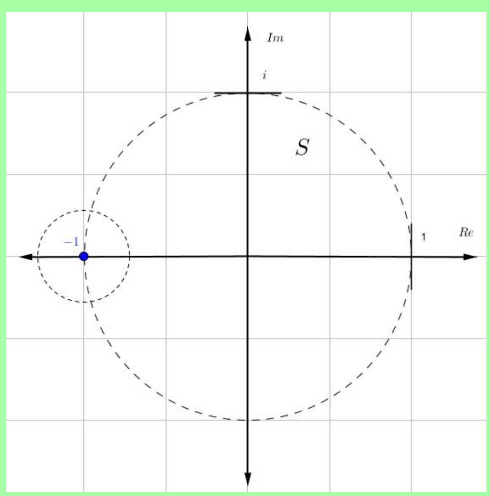
Proof. We want to choose an r such that the disk $|z - 0.9i| < r$ is completely contained in S. For example this is satisfied if we choose any $r \leq 0.1$. For example a valid choice of r would be $r = 0.05$:



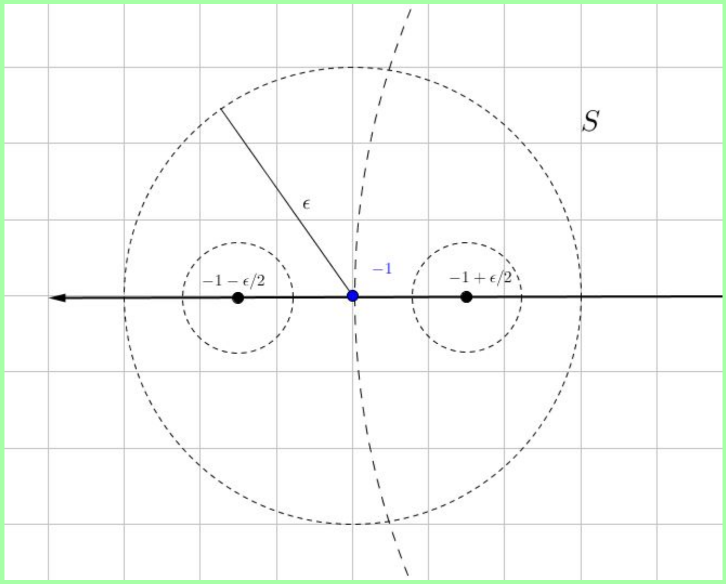
□

Example 3.7. *Prove that* -1 *is a boundary point on* S *as it was defined in the previous example.*

Proof. Consider the following diagram



Focusing in on -1 we define the points $-1 - \epsilon/2$ and $-1 + \epsilon/2$. Notice that these two new points are always in an the neighborhood of -1 with radius ϵ. Also since $|-1-\epsilon/2| > 1$ it follows that $-1-\epsilon/2 \notin S$. Similarly since $|-1+\epsilon/2|<1$ it follows that $-1+\epsilon/2\in S$. This shows that -1 is a boundary point.



□

4. Lecture 4: Regions

More Regions:

Definition 4.1. A set is **open** if all its points are interior.

For example $|z| < 1$ is an open set.

Definition 4.2. A set is **closed** if it contains its boundary.

Note that some sets are neither open or closed. The only sets in $\mathbb{C}$ which are open and closed (clopen) are $\mathbb{C}$ and $\emptyset$ (i.e. there are no nontrivial clopen sets).

Definition 4.3. The **complement** of a space S, denoted S^c, is the set of all points not in S.

Note that the boundary of S is the same as the boundary of the complement of S:

$$\partial S = \partial S^c.$$

Also the compliment of the compliment of a space S is just S. In other-words $(S^c)^c = S$.

Theorem 4.4. *S is closed $\Leftrightarrow$ S^c is open.*

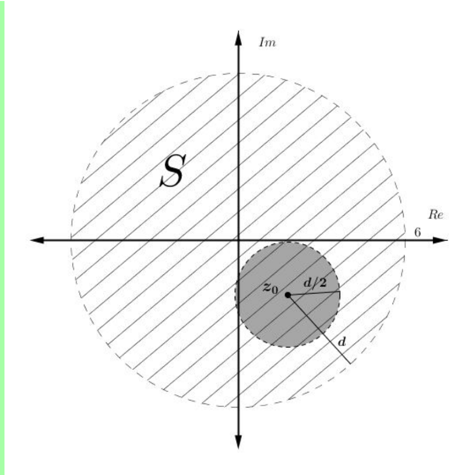
Proof. Assume S is closed.
$\Leftrightarrow$ S contains $\partial S \Rightarrow S$ contains ∂S^c.
$\Leftrightarrow$ S^c does not contain ∂S^c
$\Leftrightarrow$ All points in S^c are interior
$\Leftrightarrow$ S^c is open. □

Example 4.5. *Show that $S = \{|z| < 6\}$ is open.*
We need to check that all points in S are interior points.

Proof. Let z_0 be some point in S. So $|z_0| < 6$. Letting d be the distance from z_0 to the nearest boundary we have that $d = 6 - |z_0|$. Thus it is easy to see that the neighborhood

$$|z - z_0| < d/2$$

is completely in S. (Notice that $d/2$ is halfway between z_0 and the nearest boundary point). Since z_0 was arbitrary, all points in S are interior points which means that S is open. □

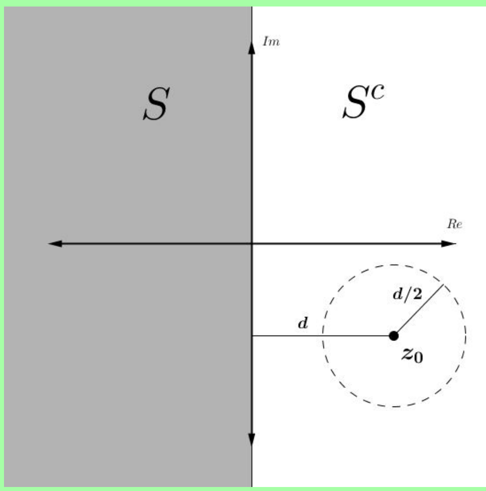


We will now do an example of showing a space it closed:

Example 4.6. *Show* $S = \{z : \mathrm{Re}(z) \leq 0\}$ *is closed.*

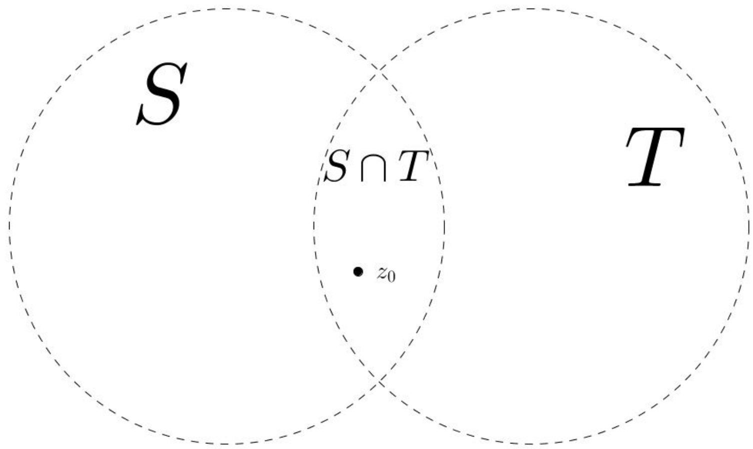
The strategry will be to show S^c *is open and then apply theorem 4.4.*

Proof. Let $z_0 \in S^c$, which means $\mathrm{Re}(z_0) > 0$. We define d to be $d := \mathrm{Re}(z_0)$. We now use $|z - z_0| < d/2$ as our neighborhood, which is contained inS^c. Since z_0 was arbitrary all points in S^c are interior which implies that S^c is open and so, by theorem 4.4, we have that S is closed. □



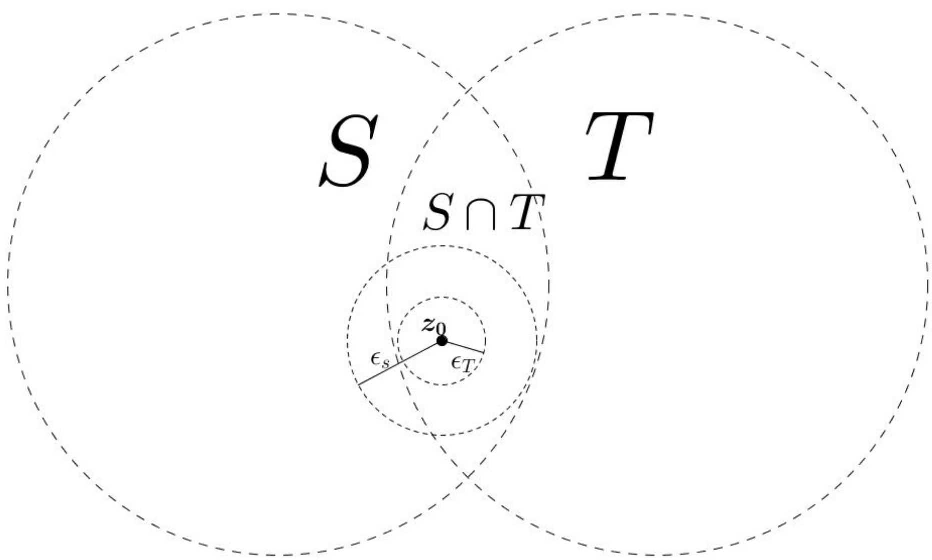
Theorem 4.7. *The intersection of 2 open sets is open.*

Proof. Let S and T be open sets.



Case1: If $S \cap T = \emptyset$, then $S \cap T$ is trivially open.

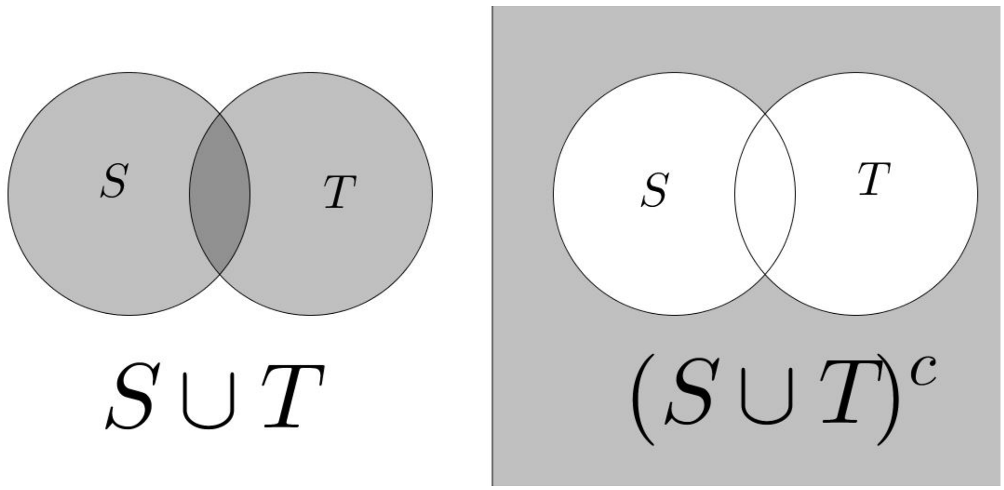
Case2: If $S \cap T \neq \emptyset$, then let $z_0 \in S \cap T$. This means that $z_0 \in S$ and $_0 \in T$. Since z_0 is an interior point of S we know that there exists a neighborhood of z_0, namely $|z - z_0| < \epsilon_S$, which is contained is S. By the same logic we know that there is a neighborhood of z_0, namely $|z - z_0| < \epsilon_T$, which is contained in T.



Let $\epsilon = \min\{\epsilon_S, \epsilon_T\}$. Notice then that $|z - z_0| < \epsilon$ is contained in both S and T (i.e. in $S \cap T$). This means that z_0 is an interior point of $S \cap T$ and since z_0 was arbitrary we have that $S \cap T$ is open. □

Theorem 4.8. *The union of 2 closed sets is closed.*

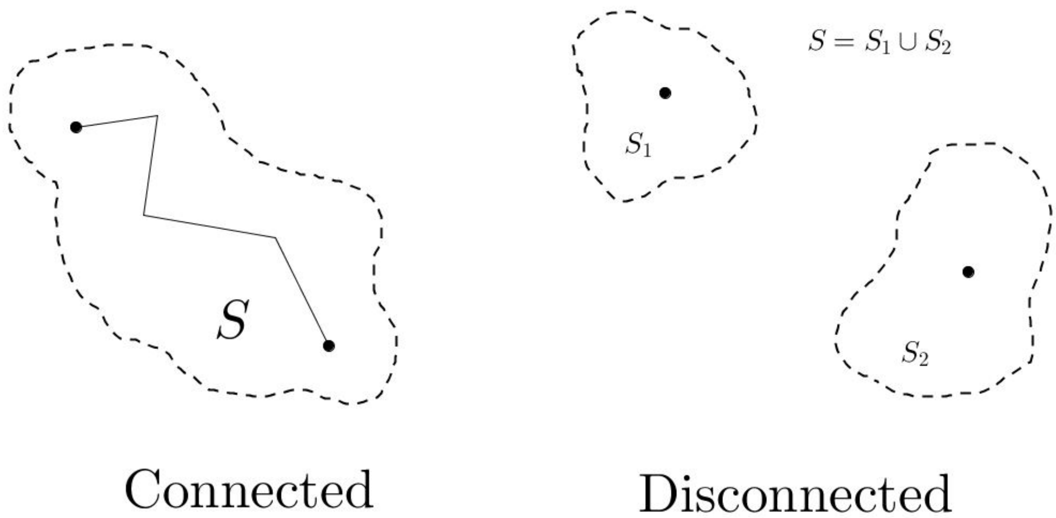
Proof. Let S and T be closed sets. To prove that $S \cup T$ is closed we will show that $(S \cup T)^c$ is open.



By theorem 4.4 S^c and T^c are both open. So applying theorem 4.7 we see that $S^c \cap T^c$ is also open. Also we know that $(S \cup T)^c = S^c \cap T^c$, so $(S \cup T)^c$ is open as well. It follows from theorem 4.4 that $S \cup T$ is closed. $\square$

5. Lecture 5: Spaces, Functions, and Mappings

Definition 5.1. An open set is **connected** if every 2 points in the set can be connected by a 'polygonal' line (i.e. a piecewise linear line) contained completely in the set.



Definition 5.2. A **domain** is a set which is open, connected, and non-empty.

Domains are nice because theorems from calculus usually carry over from $\mathbb{R}$ to domains of $\mathbb{C}$.

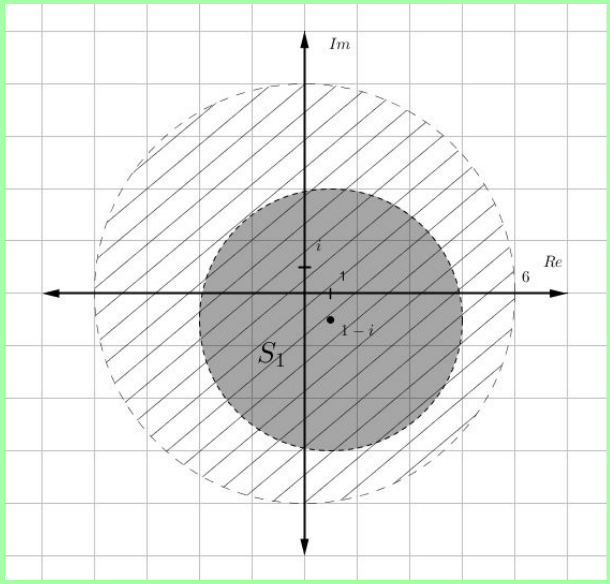
Definition 5.3. S is **bounded** if you can draw a circle around it. That is, we can find some radius R such that $S \subset \{|z| : |z| < R\}$.

Definition 5.4. If S is not bounded then it is **unbounded**.

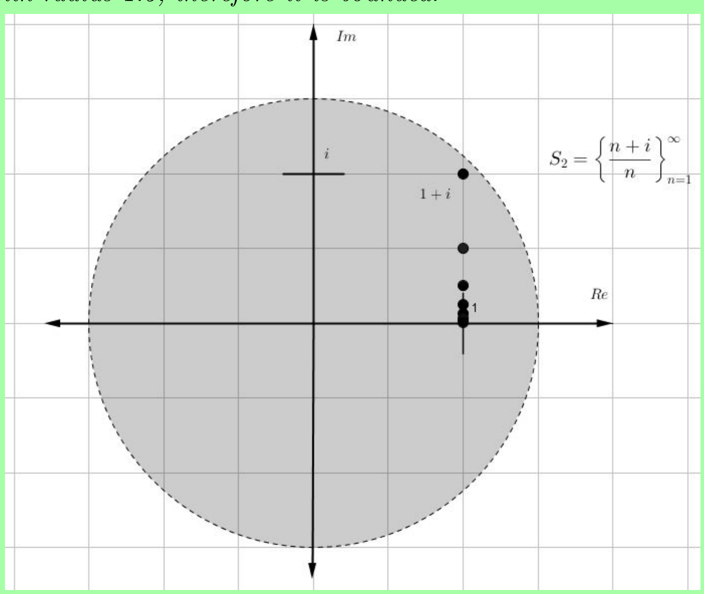
Examples 5.5. *Define the following three spaces:*

$$S_1 := |z - (i-1)| < 5$$
$$S_2 := \left\{\frac{n+i}{n}\right\}_{n=1}^{\infty} = \left\{\frac{1+i}{1}, \frac{2+i}{2}, \frac{3+i}{3}, \ldots\right\}$$
$$S_3 := \mathbb{R} = \text{ The real line.}$$

Notice S_1 is contained in the origin centered open disk with radius 6, thus S_1 is bounded.



For S_2 we see that it is entirely contained in the origin centered open disk with radius 1.5, therefore it is bounded.



Since S_3 is the real line it is impossible to draw a disk which completely contains it. Therefore S_3 in unbounded.

Definition 5.6. z_0 is an **accumulation point** of S if every neighborhood of z_0 contains infinitely many points of S.

We give an alternate definition of closed:

Definition 5.7. S is **closed** if it contains all of its accumulation points.

Functions:

A function f of z can be written $z \to f(z)$. One thing that we liked to do with real valued functions was to graph them in order to visualize the function and gain intuition. Unfortunately since our functions are from $\mathbb{C}$ (a 2-dimensional space) to $\mathbb{C}$ (a 2-dimensional space), we would need 4-dimensions to properly visualize these function (which is not possible). Despite this however, there are some things we can still do to get an intuition of what the function is doing.

Example 5.8. *Consider the function* $f(x) = z^2$.

We can split this function into two real valued functions (one for the real part and one for the imaginary part):

$$\begin{aligned} f(x+iy) &= (x+iy)^2 \\ &= (x^2 - y^2) + i(2xy). \end{aligned}$$

Thus we have split $f(z)$ *into the two real valued functions* $u(x, y) = x^2 - y^2$ *and* $v(x, y) = 2xy$. *In particular notice*

$$f(z) = f(x + iy) = u(x, y) + i \cdot v(x, y).$$

In $\mathbb{C}$ polynomials are defined as expected:

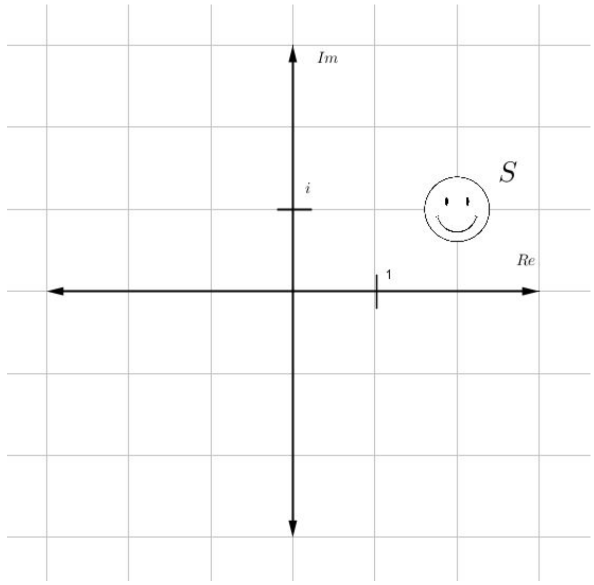
Definition 5.9. A **polynomial** with degree n is

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

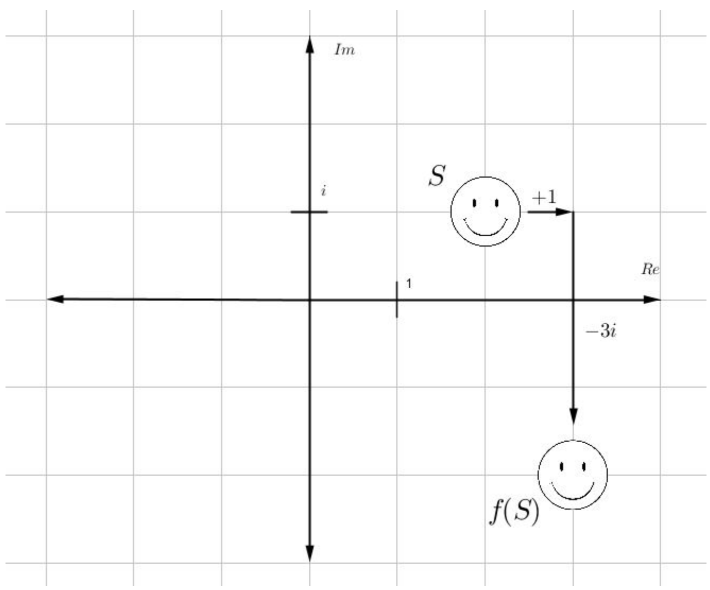
for $a_k \in \mathbb{C}$.

Mapping:

Mappings helps us to think of complex functions as moving points. We will illustrate this with several examples. In each of these examples we will take a function and see how it maps the 'smiley-face' set centered at $2 + i$. This 'smiley-face' set centered at $2 + i$ (which we will denote S) is illustrated in this diagram:



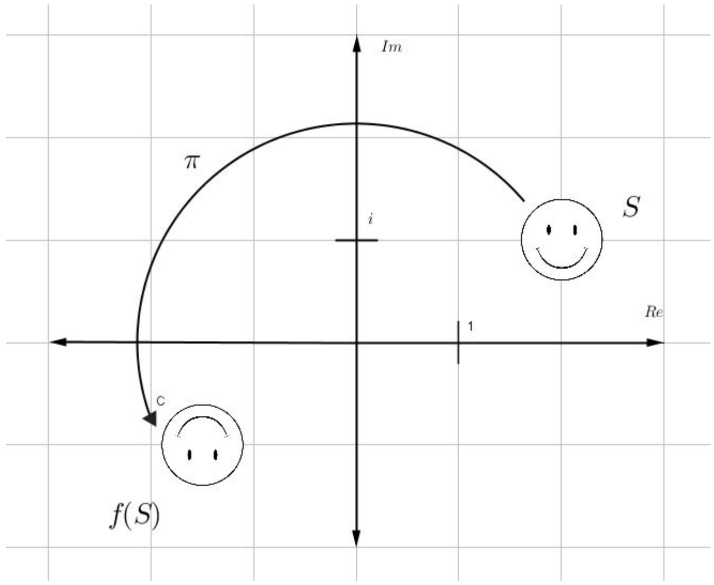
For our first example lets consider the map induced by the function $f(z) = z + 1 - 3i$. Notice that this map is just a translation, shifting S 1 unit to the right and 3 units down. So $f(S)$ with be the smiley-face centered at $3 - 2i$:



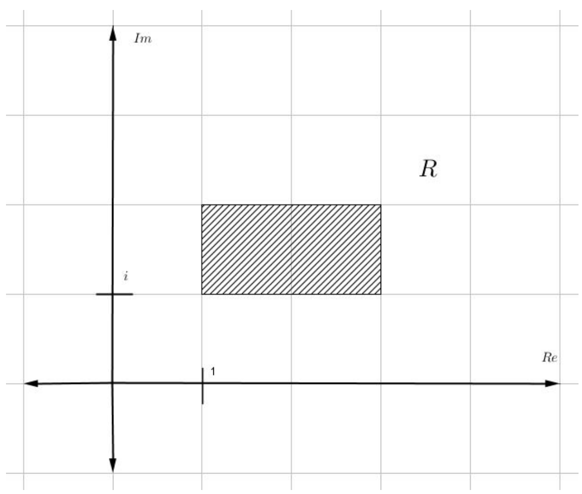
Now we will consider the function $g(z) = ze^{i\pi}$. Writing z in exponential form we have

$$\begin{aligned} g(z) &= ze^{i\pi} \\ &= re^{i(\theta+\pi)}. \end{aligned}$$

Now we see that $g(z)$ is a rotation by π radians:



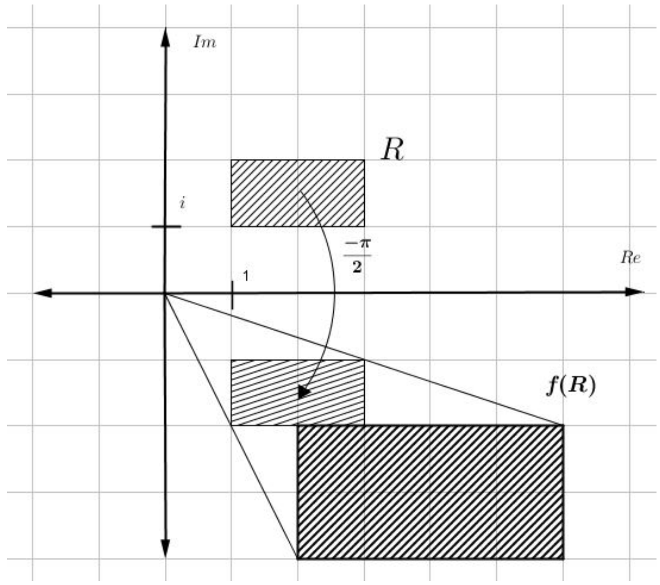
Now we will do some more mappings with a different set. The set we will know use the rectangular region $R := \{z : 1 \leq \text{Re}(z) \leq 2, 1 \leq \text{Im}(z) \leq 2$.



Now lets think about how the function $f(z) = 2ze^{i\pi/2}$ affects our region R. Putting z into exponential form we have:

$$f(z) = 2re^{i\theta} \cdot e^{i\pi/2} = 2re^{i(\theta - \pi/2)}.$$

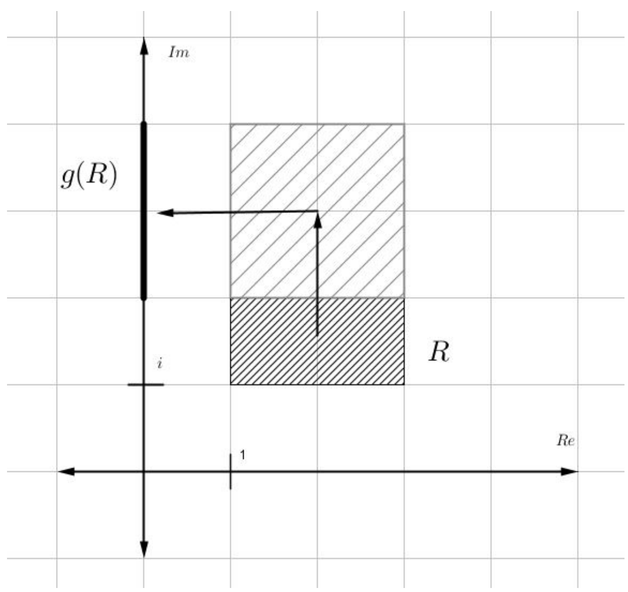
We now see that $f(z)$ will rotate all points by $-\pi/2$ radians and then double its distance from the origin. The following diagram illustrates the mapping of R to $f(R)$.



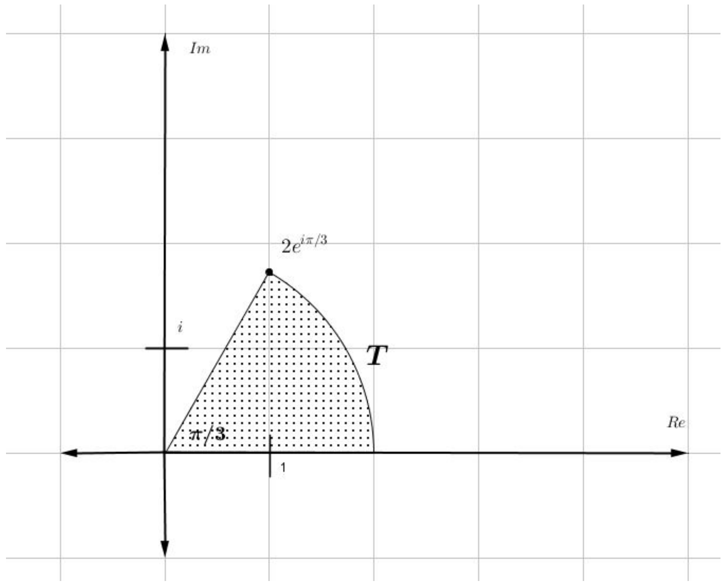
Lets consider another function, $g(z) = z - \overline{z}$. To better understand what this function is doing we make the substitution $z = x + iy$:

$$g(z) = x + iy - (x - iy) = 2iy = 2i\text{Im}(z).$$

This shows that $g(z)$ is the function which doubles the imaginary part and then makes the real part zero. The affect $g(z)$ has on R is illustrated in the following figure.



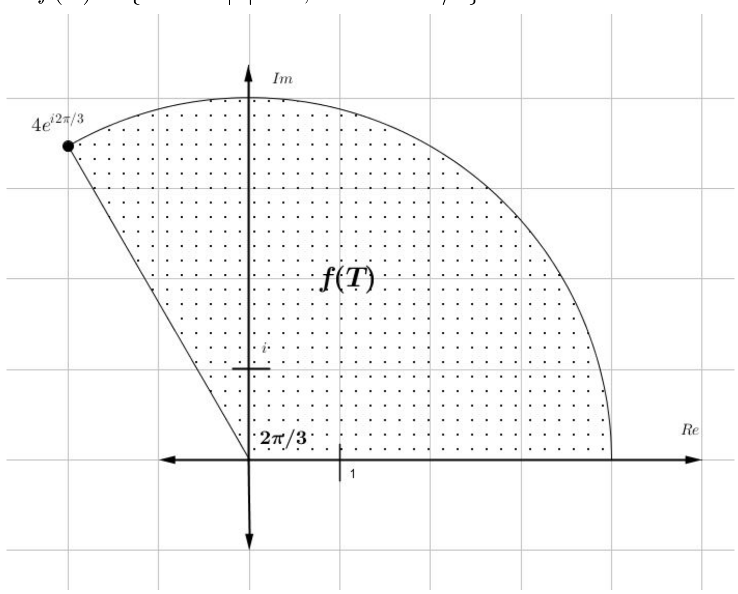
Now we will consider a couple of functions on one more region, $T := \{z : 0 < |z| < 2, 0 < \theta < \pi/3\}$:



First we consider the function $f(z) = z^2$. Taking powers suggests that we should write z exponentially;

$$f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}.$$

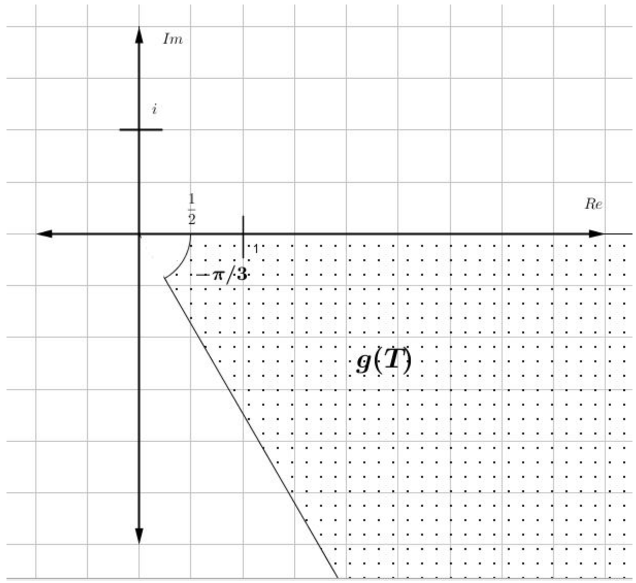
We now see that f squares the radius and double the argument. Thus the image of T will be $f(T) = \{z : 0 < |z| < 4, 0 < \theta < 2\pi/3\}$:



Now we consider the function $g(z) = 1/z$. Again working with the exponential form of z we can obtain a better understanding of this function:

$$g(z) = z^{-1} = r^{-1}e^{-i\theta}.$$

Conceptually we can think of this function as inverting the radius and flipping over the real line. We illustrate $g(T)$ in a diagram:

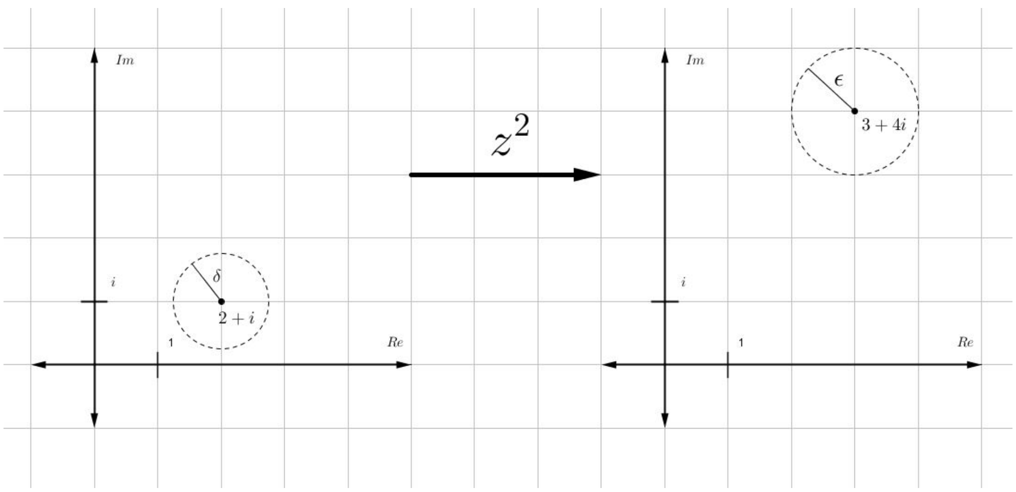

Im
i
1
2
Re
1
−π/3
g(T)

6. Lecture 6: Limits

We know that $(2+i)^2 = 3+4i$. However, what does $\lim_{z\to 2+i} z^2 = 3+4i$ mean?

Conceptually this means that if z is close to $2+i$, then z^2 is close to $3+4i$. An even better way to think of this is that the closer z is to $2+i$, the closer z^2 will be to $3+4i$.

If we know how close we need z^2 to $3+4i$, call this error ϵ, then we can say how close z must be to $2+i$ to guarantee error $< \epsilon$. We can find a neighborhood of $2+i$, (with radius δ) so that every z in the neighborhood gives $|z^2-(3+4i)| < \epsilon$.

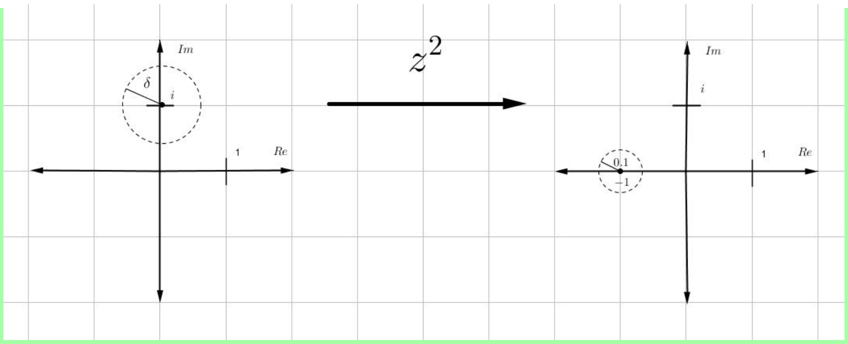


More formally we can say that $\lim_{z\to 2+i} z^2 = 3+4i$ means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $|z-(2+i)| < \delta$, then $|z^2-(3+4i)| < \epsilon$.

In general we have the following definition of a limit:

Definition 6.1. $\lim_{z\to 2+i} f(z) = L$ means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $|z-z_0| < \delta$, then $|f(z)-L| < \epsilon$.

Example 6.2. *Prove* $\lim_{z\to i} z^2 = -1$.
Start with $\epsilon = 0.1$.

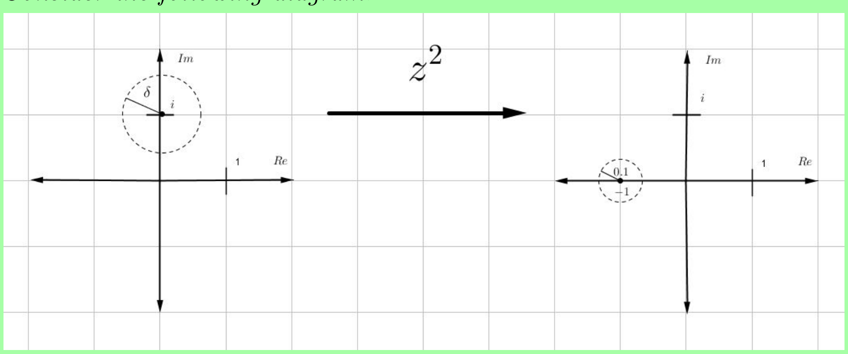


We would like to find a δ *so that any point in the neighborhood* $|z-i|<\delta$ *will be mapped to a point in the neighborhood* $|z-(-1)|<0.1$. *We see that choosing* $\delta=0.1$ *doesn't work since, for example,* $z=1.09i$ *is in the disk* $|z-i|<0.1$, *but* $(1.09i)^2=-1.1881$ *is not within a* 0.1 *neighborhood around -1.*

This means that our value for δ *must be smaller than* 0.1. *To find a value for* δ *that works trial and error is not so great; instead we will use algebra.*

$$\begin{aligned} |z^2-(-1)| &= |(z-i)(z+i)| \\ &= |z-i|\cdot|z+i| \\ &< \delta|z+i| \end{aligned}$$

Consider the following diagram:



from this we see that $|z+i|<2+\delta$ *since* z *is an arbitrary point in* $|z-i|<\delta$. *Thus we have*

$$|z^2-(-1)|<\delta|z+i|<\delta(\delta+2)<\epsilon=0.1.$$

We want to find a δ *satisfying the above inequality. We know* δ *is suppose to be small, probably less than 1 (which we can confirm by our attempt with* $\delta=0.1$*). So if* $\delta\le 1$, *then* $\delta(\delta+2)\le\delta\cdot 3$. *If* δ *is also less than or equal to* $\epsilon/3$, *then*

$$\delta(\delta+2)\le\delta\cdot 3\le\frac{\epsilon}{3}\cdot 3=\epsilon.$$

So we see that when $\epsilon=0.1$, *then* $\delta=0.1/3$ *should be sufficient.*

We finish this example by writing a more streamlined solution.

Start in the neighborhood $|z-i|<0.1/3$. Check the target distance,

$$\begin{aligned} |z^2-(-1)| &= |z+i|\cdot|z-i| \\ &< |z+i|\cdot\frac{0.1}{3} \\ &< (2+\frac{0.1}{3})\cdot\frac{0.1}{3} \\ &< 3\cdot\frac{0.1}{3} \\ &= 0.1=\epsilon. \end{aligned}$$

Thus we see that choosing $\delta=\min\{1,\epsilon/3\}$ will satisfy our limit definition.

Example 6.3. *Show* $\lim_{z\to-1}\frac{1}{2\overline{z}=1}=-1$.

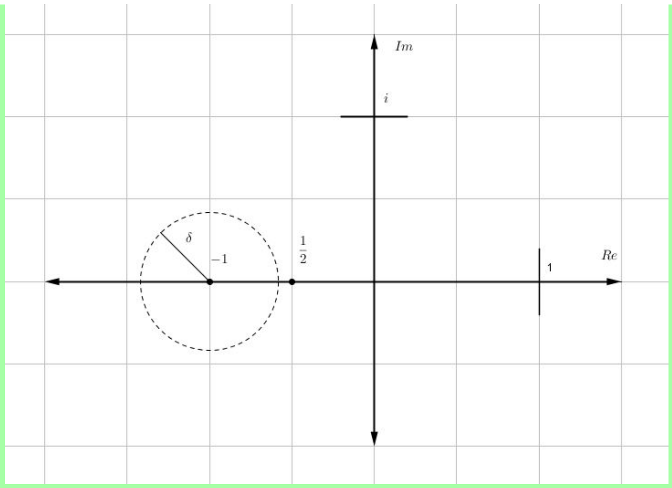
Let $\epsilon>0$ be given. If $|z-(-1)|<\delta$, then

$$\begin{aligned} |\frac{1}{2\overline{z}+1}-(-1)| &= |\frac{1}{2\overline{z}+1}+1| \\ &= |\frac{2\overline{z}+2}{2\overline{z}+1}| \\ &= |\frac{2(\overline{z}+1)}{2\overline{z}+1}| \\ &= |\frac{\overline{z}+1}{\overline{z}+1/2}| \\ &= \frac{|\overline{z}+1|}{|\overline{z}+1/2|} \\ &= \frac{|z+1|}{|z+1/2|}. \end{aligned}$$

Notice that we have gotten $|z-(-1)|$ in the numerator and we know $|z-(-1)|<\delta$, thus

$$\frac{|z+1|}{|z+1/2|}<\frac{\delta}{|\overline{z}+1/2|}.$$

We now use the following diagram to find an inequality for $|\overline{z}+1/2|$ involving δ:



In the diagram since z must lie in the disk $|z+1| < \delta$, we can see that the closest $-1/2$ can be to z is $1/2-\delta$ and the furthest $-1/2$ can be from z is $1/2+\delta$, in other words

$$\frac{1}{2} - \delta < |z + \frac{1}{2}| < \frac{1}{2} + \delta.$$

Using this we see that

$$\frac{\delta}{|z+1/2|} < \frac{\delta}{1/2-\delta}.$$

Now if $\delta \leq 1/4$ then

$$\frac{\delta}{|z+1/2|} < \frac{\delta}{1/2-1/4} = 4\delta.$$

Putting this altogether we have shown that if $\delta \leq 1/4$, then for any ϵ we can have $4\delta \leq \epsilon$ so $\delta \leq \epsilon/4$. This means if $|z+1| \leq \min\{1/4, \epsilon/4\}$, then

$$|\frac{1}{2\overline{z}+1} + 1| < \epsilon,$$

which is what we wanted to show.

We now give an example where the limit is NOT true.

Example 6.4. *Try to show* $\lim_{z\to 4} 2z - 3i = 6 - 3i$.

Notice that when $z = 4$ we have $2z - 3i = 8 - 3i$. Since $8 - 3i \neq 6 - 3i$ we see that when ϵ is appropriately small $8 - 3i$ will be outside the disk $|z - (6 - 3i)| < \epsilon$, and thus there is no value of δ that will work (since regardless of how small δ is $2z - 3i$ will still be in the starting disk).

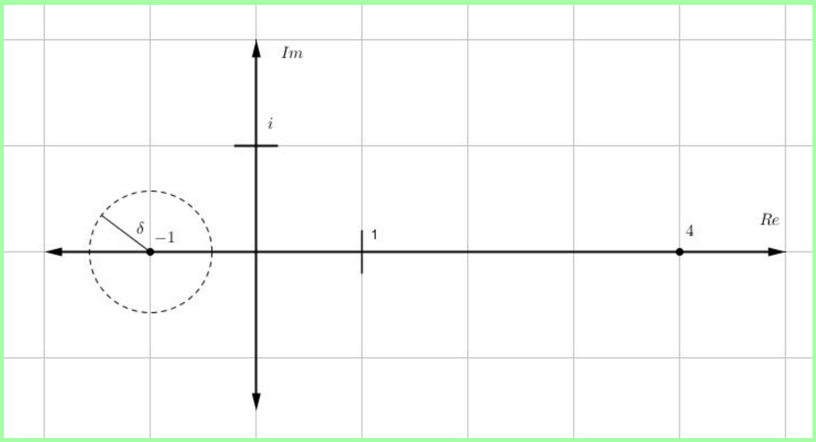
We give one final example.

Example 6.5. *Prove* $\lim_{z\to -1} z^2 - 3z + 2 = 6$.

Proof. Let $\epsilon > 0$ be given. If $|z+1| < \delta$, then

$$\begin{aligned} |z^2 - 3z + 2 - 6| &= |z^2 - 3z - 4| \\ &= |z-4| \cdot |z+1|. \end{aligned}$$

But we know $|z-4| \cdot |z+1| < |z-4|\delta$. Also, using the following diagram we can get a bound for $|z-4|$ in terms of δ.



From this we see that $5 - \delta < |z-4| < 5 + \delta$. Also, from this, if we make $\delta \le 1$ then we have that $4 < |z-4| < 6$. So now we have

$$|z-4|\delta < 6\delta \le \epsilon.$$

And thus $\delta \le \epsilon/6$.

More succinctly we have for any given ϵ:

If $|z+1| < \delta$ *where* $\delta = \min\{1, \epsilon/6\}$,
then $|(z^2 - 3z + 2) - 6| < \epsilon$.

□

7. Lecture 7: Limits That Do Not Exist and Infinity

Limits do not always exist. For example the limit

$$\lim_{z\to 0}\frac{z}{\overline{z}}$$

Does NOT Exist (DNE). In general to show the existence of a limit we try to land everything into a target neighborhood of L with radius ϵ. To show limit does not exist we want to show that points are 'sent to for far away places'. Lets revisit the limit we opened this section with:

Example 7.1. *Show*

$$\lim_{z\to 0}\frac{z}{\overline{z}}$$

does not exists.

Let us write z as $x+iy$ so that

$$\frac{z}{\overline{z}}=\frac{x+iy}{x-iy}.$$

For complex limits in order for the limit to exist all limits (from all directions) must agree. Thus if we find that the limit from two different directions dissagree we will have shown that the limit does not example. For our particular example we will consider the limit from the real and imaginary axi.

From Real Axis. *By fixing $y=0$ and considering the limit as x approaches zero we get the limit as we approach along the real axis. The limit is*

$$\lim_{(x,0)\to(0,0)}\frac{x+0}{x-0}=1.$$

From Imaginary Axis. *Conversely if we fix $x=0$ and consider the limit as y approaches zero we will obtain a limit as we approach $(0,0)$ from the imaginary axis:*

$$\lim_{(0,y)\to(0,0)}\frac{0+iy}{0-iy}=\lim_{(0,y)\to(0,0)}\frac{-y}{y}=-1$$

So the 'real' limit and the 'imaginary' limits disagree ($1\neq -1$), thus the limit does not exist.

Example 7.2. *Show*

$$\lim_{z\to 0}\frac{z^2}{|z|^2}$$

does not exist.

Fistly let us rewrite $z=x+iy$, to obtain

$$\lim_{z\to 0}\frac{z^2}{|z|^2}=\lim_{(x,y)\to(0,0)}\frac{(x+iy)^2}{x^2+y^2}.$$

If we fix $y = 0$ we get the limit

$$\lim_{(x,0)\to(0,0)} \frac{(x+0)^2}{x^2+0^2} = 1.$$

Conversely if we fix $x = 0$ we get the limit

$$\lim_{(0,y)\to(0,0)} \frac{(0+iy)^2}{0+y^2} = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1.$$

Since $1 \neq -1$ we see that the limit does not exist.

One may have observed that the 'two' examples we have given are really the same since

$$\frac{z^2}{|z|^2} = \frac{z^2}{z\overline{z}} = \frac{z}{\overline{z}}.$$

We proceed with a new example (which is actually different from the previous two):

Example 7.3. *Show the limit*

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)\mathrm{Im}(z)}{|z^2|}$$

does not exist.

Firstly we make the substitution $z = x + iy$:

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)\mathrm{Im}(z)}{|z^2|} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

If we fix $y = 0$ then

$$\lim_{(x,0)\to(0,0)} \frac{x0}{x^2+0^2} = 0.$$

Similarly by fixing $x = 0$ we have

$$\lim_{(0,y)\to(0,0)} \frac{0y}{0^2+y^2} = 0.$$

So far the limits from the real and imaginary axis agree, so we can say nothing about the limits existence. However lets consider coming in from another direction and see if the limit still agrees.

By fixing $x = y = c$ our limit becomes

$$\lim_{(c,c)\to(0,0)} \frac{c^2}{2c^2} = \frac{1}{2}.$$

But now, since $1/2 \neq 0$ the limit does not exist.

We state some useful theorems about limits:

Theorem 7.4. *You can take the limit of the real and imaginary parts separately.*

For example if we have a function $f(z) = u(x, y) + i \cdot v(x, y)$ for $z = x + iy$. Additionally if $f(z)$ approaches the limit $L = u_0 + v_0$ as z approaches $z_0 = x_0 + iy_0$

then we have the following:

$$\lim_{z\to z_0} f(z) = L \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \qquad \textit{and} \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Theorem 7.5. *If*

$$\lim_{z\to z_0} f_1(z) = L_1 \qquad \textit{and} \qquad \lim_{z\to z_0} f_2(z) = L_2$$

then

$$\lim_{z\to z_0} (f_1(z) + f_2(z)) = L_1 + L_2$$

Theorem 7.6. *If*

$$\lim_{z\to z_0} f_1(z) = L_1 \qquad \textit{and} \qquad \lim_{z\to z_0} f_2(z) = L_2$$

then

$$\lim_{z\to z_0} (f_1(z) \cdot f_2(z)) = L_1 \cdot L_2.$$

Proof. Assume we know

$$\lim_{z\to z_0} f_1(z) = L_1 \qquad \textit{and} \qquad \lim_{z\to z_0} f_2(z) = L_2.$$

This means to "'hit an L_1 target"' or "'L_2 target"' respectively. In other words given a we know there exists a δ_1 and a δ_2 which satisfy $|z - z_0| < \delta_1$ implies $|f_1(z) - L_1| < \epsilon_1$ and $|z - z_0| < \delta_2$ implies $|f_2(z) - L_2| < \epsilon_2$ for any given ϵ_1 and ϵ_2.

So let $\epsilon > 0$ be given. We need to find a δ satisfying $|z - z_0| < \delta$ which will imply $|f_1(z)f_2(z) - L_1L_2| < \epsilon$. Rewriting this we have

$$|f_1(z)f_2(z) - L_1L_2| = |f_1(z)f_2(z) - L_1f_2(z) + L_1f_2(z) - L_1L_2|$$

Now by the triangle inequality

$$|f_1(z)f_2(z) - L_1f_2(z) + L_1f_2(z) - L_1L_2| \le |f_2(z)(f_1(z) - L_1)| + |L_1(f_2(z) - L_2|$$

and note

$$|f_2(z)(f_1(z) - L_1)| + |L_1(f_2(z) - L_2| = |f_2(z)||f_1(z) - L_1| + |L_1||f_2(z) - L_2|.$$

Now if $|f_2(z) - L_2| \le (1/2)(\epsilon/|L_1|)$ we would be happy since we would have

$$|f_2(z)||f_1(z) - L_1| + |L_1||f_2(z) - L_2| < |f_2(z)||f_1(z) - L_1| + \epsilon/2$$

and we would be well on our way to finding a suitable δ. The good new is, is that we can in-fact do this by letting $\epsilon_2 = \frac{1}{2}\frac{\epsilon}{|L_1|}$ we can find a δ_2.

If $\epsilon_2 \le 1$ then $|f)2(z)| \le |L_2| + 1$. And so

$$|f_2(z)||f_1(z) - L_1| + |L_1||f_2(z) - L_2| \le (|L_2| + 1)|f_1(z) - L_1| + |L_1||f_2(z) - L_2|.$$

Now letting $f_1(z) - L_1 < \epsilon_1 = (1/2)(\epsilon/(|L_1| + 1))$, we have

$$|L_2| + 1)|f_1(z) - L_1| + |L_1||f_2(z) - L_2| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Now just use the δ which satisfies all 3 conditions. □

Theorem 7.7. *If*

$$\lim_{z\to z_0} f_1(z) = L_1 \qquad \textit{and} \qquad \lim_{z\to z_0} f_2(z) = L_2$$

then

$$\lim_{z\to z_0} (f_1(z)/f_2(z)) = L_1/L_2$$

so long that $L_2 \neq 0$.

Theorem 7.8. *If* $P(z)$ *is a polynomial then*

$$\lim_{z\to z_0} P(z) = P(z_0).$$

Proof. If $P(z) = c$, for $c \in \mathbb{C}$ then $\lim_{z\to z_0} c = c$. This can easily be proven with an $\epsilon - \delta$ proof.

We also have $\lim_{z\to z_0} z = z_0$ which can be again proven easily with an $\epsilon - \delta$ proof.

Now our result follows easily by theorem 7.5 and theorem 7.6. (Recal that $P(z)$ can be written $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ for $a_i \in \mathbb{C}$. □

Theorem 7.9. *If* $P(z)$ *and* $Q(z)$ *are polynomials and* $Q(z_0) \neq 0$ *then*

$$\lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0}.$$

Infinity

We have the following way to think of infinity

$$z \to \infty \Leftrightarrow \frac{1}{z} \to 0,$$

and

$$\lim_{z\to z_0} f(z) = \infty \Leftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

In-fact we have the equivalence

$$\lim_{z\to\infty} f(z) = \lim_{z\to z_0} \frac{1}{z}.$$

And from this

$$\lim_{z\to\infty} f(z) = \infty \Leftrightarrow \lim_{z\to z_0} \frac{1}{f(1/z)} = 0?.$$

Example 7.10. *Find the following limit:*

$$\lim_{z\to\infty} \frac{1}{z+1}.$$

We have that

$$\lim_{z\to\infty}\frac{1}{z+1} = \lim_{z\to 0}\frac{1}{1/z+1} = \lim_{z\to 0}\frac{z}{1+z} = 0.$$

Example 7.11. *Find the following limit:*

$$\lim_{z\to\infty}\frac{z^2+4}{z^2-4}.$$

We have that

$$\lim_{z\to\infty}\frac{z^2+4}{z^2-4} = \lim_{z\to 0}\frac{(1/z)^2+4}{(1/z)^2-4} = \lim_{z\to 0}\frac{1+4z^2}{1-z} = 1.$$

Example 7.12. *Find*

$$\lim_{z\to 0}\frac{1}{z^2}.$$

Because

$$\lim_{z\to 0}\frac{1}{(1/z)^2} = \lim_{z\to 0} z^2 = 0$$

we also have

$$\lim_{z\to 0}\frac{1}{z^2} = \infty.$$

Example 7.13. *Find the limit*

$$\lim_{z\to -1}\frac{iz+3}{z+1}.$$

Because

$$\lim_{z\to -1}\frac{z+1}{iz+3} = \frac{0}{3} = 0$$

we have

$$\lim_{z\to -1}\frac{iz+3}{z+1} = \infty.$$

Example 7.14. *Find the limit*

$$\lim_{z\to 0}\frac{z^2+3z}{z-2}.$$

This is straightforward

$$\lim_{z\to 0} \frac{z^2+3z}{z-2} = \frac{0^2+3\cdot 0}{0-2} = 0.$$

Example 7.15. *Find the limit*

$$\lim_{z\to\infty} \frac{2z^3-1}{z^2+1}.$$

We have

$$\begin{aligned}\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} &= \lim_{z\to 0} \frac{2(1/z)^3-1}{(1/z)^2+1} \\ &= \lim_{z\to 0} \frac{2-z^3}{z+z^3}.\end{aligned}$$

Since

$$\lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0,$$

we know

$$\lim_{z\to -1} \frac{iz+3}{z+1}.$$

Example 7.16. *Find the limit*

$$\lim_{z\to\infty} \frac{2z+i}{z+1}.$$

We have

$$\begin{aligned}\lim_{z\to\infty} \frac{2z+i}{z+1} &= \lim_{z\to 0} \frac{2(1/z)+i}{(1/z)+1} \\ &= \lim_{z\to 0} \frac{2+iz}{1+z} \\ &= \frac{2}{1} = 2.\end{aligned}$$

8. Lecture 8: Derivatives

Definition 8.1. A function is **continuous** at z_0 if the following 3 conditions hold:

(1) $f(z_0)$ exists
(2) $\lim_{z\to z_0} f(z)$ exist.
(3) $\lim_{z\to z_0} f(z) = f(z_0)$.

We say a function f is continuous on a set S if f is continuous at every $z_0 \in S$. We give a formal definition of the derivative:

Definition 8.2. We define the **derivative** in the following fashion:

$$f'(z) := \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} // \qquad = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$$

Make note that unlike the real derivative this ‘complex-derivative’ does NOT represent the slope.

Lets give the example of taking a derivative using the formal definition.

Example 8.3. *Find the derivative of $f(z) = z^2$.*

$$\begin{aligned} f'(z) &= \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z\to 0} 2z + \Delta z \\ &= 2z. \end{aligned}$$

Example 8.4. *Find the derivative of $f(z) = \overline{z}$.*

$$\begin{aligned} f'(z) &= \lim_{\Delta z\to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}. \end{aligned}$$

But by lecture 7 we know that limit does not exist, and so the derivative also does not exist.

Example 8.5. *Find the derivative of $f(z) = |z|^2$.*

Notice $|z|^2 = z \cdot \overline{z}$. Using the definition of the derivative we have

$$\begin{aligned} f'(z) &= \lim_{\Delta \to z} \frac{(z+\Delta z)\cdot(\overline{z+\Delta z}) - z\cdot\overline{z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z\overline{z} + z\overline{\Delta z} + \Delta z\overline{z} + \Delta z\cdot\overline{\Delta z} - z\overline{z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z\overline{\Delta z} + \Delta z\overline{z} + \Delta z\overline{\Delta z}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{z\overline{\Delta z}}{\Delta z} + \overline{z} + \overline{\Delta z} \end{aligned}$$

Notice that from Lecture 7 this limit only exists at $z = 0$.

Many of the formulas for derivatives we are accustom to carry over to complex analysis:

$$\begin{aligned} &\frac{d}{dz}(c) = 0 \\ &\frac{d}{dz}(z) = 1 \\ &\frac{d}{dz}(z^n) = nz^{n-1} \\ &\frac{d}{dz}(c\cdot f(z)) = c\cdot f'(z) \\ &\frac{d}{dz}(f(z) + g(z) = f'(z) + g'(z). \end{aligned}$$

The product, quotient, and chain rules all work as well. For the most part these can be proved in the same way as they were in the real case.

We would like to obtain a way of checking whether or not a given function is differentiable. Recall

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

We can change this to real functions by letting $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$,

$$f'(z) = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i(v(x+\Delta x, y+\Delta y) - v(x,y))}{\Delta x + i\Delta y}.$$

Lets check what occurs when $\Delta y = 0$ and what happens when $\Delta x = 0$; if these give different limits then the limit does not exist (so neither does the derivative).

(1) Let $\Delta y = 0$:

$$\begin{aligned} f'(z) &= \lim_{(\Delta x,0)\to(0,0)} \frac{u(x+\Delta x, y) - u(x,y) + i(v(x+\Delta x, y) - v(x,y))}{\Delta x} \\ &= \lim_{(\Delta x,0)\to(0,0)} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{i(v(x+\Delta x, y) - v(x,y))}{\Delta x} \\ &= U_x + iV_x, \end{aligned}$$

where U_x is the partial derivative of $u(x, y)$ with respect to x and V_x is the partial derivative of $v(x, y)$ with respect to x.

(2) Let $\Delta x = 0$:

$$\begin{aligned} f'(z) &= \lim_{(0,\Delta y)\to(0,0)} \frac{u(x, y + \Delta y) - u(x, y) + i(v(x, y + \Delta y) - v(x, y))}{i\Delta y} \\ &= \lim_{(0,\Delta y)\to(0,0)} \frac{-i(u(x, y + \Delta y) - u(x, y))}{\Delta y} + \frac{(v(x, y + \Delta y) - v(x, y))}{\Delta y} \\ &= -iU_y = V_y, \end{aligned}$$

where U_y and V_y are the respective partial derivatives of $u(x, y)$ and $v(x, y)$ in terms of y.

So, in order for the limit to exist we must have

$$U_x + iV_x = V_y - iU_y.$$

If we equate real and imaginary parts we obtain

$$U_x = V_y \qquad \textit{and} \qquad V_x = -U_y.$$

The above equations are very important and are called the **Cauchy-Riemann equations**.

Theorem 8.6. *If the Cauchy Riemann equations are satisfied and the partial derivatives U_x, U_y, V_x, V_y are each continuous, then $f'(z)$ exists and is*

$$f'(z) = U_x + iV_x = V_y - iU_y.$$

Now lets repeat our earlier examples with this new theorem. For the most part we will assume the partial derivatives are continuous for the following examples.

Example 8.7. *Find the derivative of $f(z) = z^2$.*

Letting $z = x + iy$ we obtain

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Thus, $u(x, y) = x^2 + y^2$ and $v(x, y) = 2xy$. From these we can calculate the following partial derivatives:

$$\begin{aligned} U_x &= 2x \\ U_y &= -2y \\ V_x &= 2y \\ V_y &= 2x. \end{aligned}$$

We see that the Cauchy-Riemann equations are satisfied since $2x = 2x$ and $-2y = -(2y)$. Moreover

$$f'(z) = 2x + i2y = 2z.$$

Example 8.8. *Find the derivative of $f(z) = \overline{z}$.*

Let $z = x + iy$ so that we obtain

$$f(x + iy) = \overline{x + iy} = x - iy.$$

So, $u(x, y) = x$ and $v(x, y) = -y$. From these we have the partial derivatives

$$U_x = 1$$
$$U_y = 0$$
$$V_x = 0$$
$$V_y = -1.$$

Notice that the Cauchy-Riemann equations are NOT satisfied since $U_x \neq V_y$. Therefore the limit and the derivative do not exist.

Example 8.9. *Find the derivative of $f(z) = |z|^2$. Letting $z = x + iy$,*

$$f(x + iy) = x^2 + y^2 + 0i.$$

Thus, $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Taking partial derivatives we get

$$U_x = 2x$$
$$U_y = 2y$$
$$V_x = 0$$
$$V_y = 0.$$

From this we see that the Cauchy-Riemann equations are not satisfied in general, only when $x = y = 0$. Thus the derivative does not exist anywhere except for the origin. At this single point the derivative is $f'(0) = 0$.

Lets move on to some new examples.

Example 8.10. *Find the derivative of e^z.*

Letting $z = x + iy$ we can rewrite this as

$$\begin{aligned} f(z) &= e^x e^{iy} \\ &= e^x(\cos(y) + i\sin(y)). \end{aligned}$$

So $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. Computing the partial derivatives of these we get

$$U_x = e^x \cos(y)$$
$$U_y = -e^x \sin(y)$$
$$V_x = e^x \sin(y)$$
$$V_y = e^x \cos(y).$$

We see that the Cauchy-Riemann equations are satisfied for all values of x and y and that the derivative is

$$\begin{aligned} f'(z) &= e^x\cos(y) + ie^x\sin(y) \\ &= e^x(\cos(y) + i\sin(y)) \\ &= e^x e^{iy} \\ &= e^z. \end{aligned}$$

Cauchy-Riemann in Polar Coordinates

Sometimes it will be convenient to use a polor-coordinate version of the Cauchy-Riemann equations. In this case we will have

$$f(z) = u(r,\theta) + iv(r,\theta).$$

Using the change of variables $x = r\cos\theta$ and $y = r\sin\theta$ to change into polar form see that

$$u(x,y) + iv(x,y) \to u(r\cos\theta, r\sin\theta) + iv(r\cos\theta, r\sin\theta).$$

Now we want to find the partials $U_r, U_\theta, V_r, V_\theta$. To do this we will recall the chain rule from Calc III to obtain

$$\begin{aligned} U_r &= U_x X_r + U_y Y_r \\ &= U_x\cos\theta + u_y\sin\theta. \end{aligned}$$

Similarly we can find U_θ:

$$\begin{aligned} U_\theta &= U_x X_\theta + U_y Y_\theta \\ &= U_x(-r\sin\theta) + U_y(r\cos\theta). \end{aligned}$$

Analogous computations show

$$\begin{aligned} V_r &= V_x\cos\theta + V_y\sin\theta \\ V_\theta &= V_x(-r\sin\theta) + V_y(r\cos\theta). \end{aligned}$$

Using the fact that $V_x = -U_y$ and $U_x = V_y$ (from the CR equations), we have

$$\begin{aligned} V_r &= -U_y\cos\theta + U_x\sin\theta \\ V_\theta &= U_y(-r\sin\theta) + U_x(r\cos\theta). \end{aligned}$$

From the partial derivatives we've computed (possibly along with a little linear algebra) one can show that the polar Cauchy-Riemann equations are

$$V_\theta = rU_r \qquad \text{and} \qquad U_\theta = -rV_r.$$

Example 8.11. *Find the derivative of $\frac{1}{z}$ using the polar Cauchy-Riemann equations.*

Let $z = re^{i\theta}$. Then we have

$$\begin{aligned}\frac{1}{z} &= \frac{1}{re^{i\theta}} \\ &= \frac{1}{r}e^{-i\theta} \\ &= \frac{1}{r}(\cos(-\theta) + i\sin(-i\theta)) \\ &= \frac{1}{r}\cos(\theta) - \frac{1}{r}i\sin(\theta).\end{aligned}$$

Thus, $u(r, \theta) = (1/r)\cos(\theta)$ and $v(r, \theta) = (1/r)\sin(\theta)$. From these we obtain the following partial derivatives

$$\begin{aligned}U_r &= \frac{-1}{r^2}\cos(\theta) \\ U_\theta &= \frac{-1}{r}\sin(\theta) \\ V_r &= \frac{1}{r^2}\sin(\theta) \\ V_\theta &= \frac{-1}{r}\cos(\theta).\end{aligned}$$

We have $rU_r = V_\theta$ and $U_\theta = -rV_r4$, so the Cauchy-Riemann equations are satisfied and so $f'(z)$ exists for all $z \in \mathbb{C}$. In fact the derivative is

$$\begin{aligned}f'(z) &= e^{-i\theta}(U_r + iV_r) \\ &= e^{-i\theta}(\frac{-1}{r^2}\cos(\theta) + i\frac{1}{r^2}\sin(\theta)) \\ &= \frac{-e^{-i\theta}}{r^2}(\cos(\theta) - i\sin(\theta)) \\ &= \frac{-(e^{-i\theta})^2}{r^2} \\ &= \frac{-1}{(re^{i\theta})^2} \\ &= \frac{-1}{z^2}.\end{aligned}$$

9. Lecture 9: Analytic Functions and Harmonic Functions

Analytic Functions

Definition 9.1. A function f is **analytic** if it is differentiable. If f is differentiable in an open set then f is **analytic** in that set.

Remark 9.2. Analytic functions are also called **holomorphic** functions.

Definition 9.3. If f is analytic for all complex numbers, call it **entire**.

Theorem 9.4. *If $f'(z) = 0$ everywhere in domain D, then $f(z)$ must be constant in D.*

(Recall that a domain must be open and connected).

Proof. Split f into real functions u and v as $f(z) = u + iv$. Since $f'(z)$ exists, it must satisfy the Cauchy-Riemann equations. We also have that

$$f'(z) = U_x + iV_x = 0.$$

Thus, $U_x = V_x = 0$ and by the Cauchy-Riemann equations we must also have $U_y = V_y = 0$. But now we see that all the partials of u and v are 0 so each of these functions must be constant and so must $f(z) = u + iv$. □

Remark 9.5. In general we will use ln for the real logarithm and log for the complex logarithm.

Example 9.6. *Find the derivative of $f(z) = \ln(r) + i\theta$ for $0 < \theta < 2\pi$.*

We must check that $f(z)$ is analytic where it is defined. Notice that if $f(z) = u + iv$ then we have $u(r, \theta) = \ln(r)$ and $v(r, \theta) = \theta$. From these we get the following partial derivatives:

$$U_r = \frac{1}{r}$$
$$U_\theta = 0$$
$$V_r = 0$$
$$V_\theta = 1.$$

Since $rU_r = V_\theta$ and $-rV_r = U_\theta$, the Cauchy-Riemann equations are satisfied. Moreover,

$$\begin{aligned} f'(z) &= e^{-i\theta}(U_r + iV_r) \\ &= e^{-i\theta}(\frac{1}{r} + 0) \\ &= \frac{1}{z}. \end{aligned}$$

Notice that

$$\log(re^{i\theta} = \log(r) + \log(e^{i\theta}) = \ln(r) + i\theta,$$

So infact the last example shows that the derivative of $\log(z)$ is $1/z$.

Theorem 9.7. *If f and $\overline{f}$ are both analytic, then f is constant.*

Proof. Let $f(z) = u + iv$ and subsequently $\overline{f(z)} = u - iv$. Since f is analytic we must have $U_x = V_y$ and $U_y = -V_x$. Similarly, since $\overline{f}$ is analytic, we must have $U_x = -V_y$ and $U_y = V_x$. But now we have

$$U_y = -V_x \ \ \& \ \ U_y = V_x \Rightarrow 2U_y = 0 \Rightarrow U_y = 0,$$

and similarly

$$U_y = -V_x \ \ \& \ \ U_y = V_x \Rightarrow 2V_y = 0 \Rightarrow V_y = 0.$$

But now our partial derivatives are all zero. This means that u and v must be constant and so f must be constant as well. ☐

Harmonic Functions

Definition 9.8. A real-valued function $H(x, y)$ is called a **Harmonic Function** if the following two criterion are met:

(1) All second partial derivatives are continuous
(2) $H_{xx} + H_{yy} = 0$.

Remark 9.9. The equation $H_{xx} + H_{yy} = 0$ is **Laplace's equation**. This equation is important and physics. Harmonic functions can be used for problems such as steady state of heat, electric charges, etc.

Theorem 9.10. *If $f = u + iv$ is analytic in a domain D, then u and v are harmonic functions.*

Proof. One of the things we must show is that the second partial derivatives are continuous. We will hold off on showing this until subsequent lectures.

We will however show that Laplace's equation is satisfied. Since f is analytic the Cauchy-Riemann equations are satisfied and we have

$$\frac{\partial}{\partial x}(U_x = V_y) \Rightarrow U_{xx} = V_{yx}.$$

and

$$\frac{\partial}{\partial y}(U_y = -V_x) \Rightarrow U_{yy} = -V_{xy}.$$

Now, by assuming the second partials are continuous, we have $-V_{xy} = -V_{yx}$. Therefore,

$$U_{xx} + U_{yy} = V_{yx} - V_{yx} = 0.$$

A similar argument shows that $U_{yy} + V_{yy}$. Thus Laplace's equation holds. □

Consider e^{-z}, notice

$$\begin{aligned} e^{-z} &= e^{-x-iy} \\ &= e^{-x}(\cos(y) - i\sin(y)) \\ &= e^{-x}\cos(y) - ie^{-x}\sin(y). \end{aligned}$$

Recall that we have seen that e^z is an entire function. Consequently this means that $u(x,y) = e^{-x}\cos(y)$ and $v(x,y) = -e^{-x}\sin(y)$ should be harmonic functions. Postponing the process of showing that the partial derivatives are continuous we will just show that Laplace's equation is satisfied. We have the following partial derivatives:

$$\begin{aligned} U_x &= -e^{-x}\cos(y) \\ U_y &= -e^{-x}\sin(y) \\ V_x &= -e^{-x}\sin(y) \\ V_y &= e^{-x}\cos(y). \end{aligned}$$

From these we get the second partial derivatives

$$\begin{aligned} U_{xx} &= e^{-x}\cos(y) \\ U_{yy} &= -e^{-x}\cos(y) \\ V_{xx} &= e^{-x}\sin(y) \\ V_{yy} &= -e^{-x}\sin(y). \end{aligned}$$

We see that $U_{xx} + U_{yy} = V_{xx} + V_{yy} = 0$, and Largrange's equation is satisfied.

Definition 9.11. If u and v are harmonic functions and satisfy $U_x = V_y$ and $U_y = -V_x$ then $u + iv$ is analytic and we say V is a **Harmonic Conjugate** of U.

Example 9.12. *Find a harmonic conjugate of $U = 2xy$.*

First we must check that U is harmonic. Notice

$$U_{xx} = 0 \qquad \text{and} \qquad V_{yy} = 0,$$

so $U_{xx} + V_{yy} = 0$ and U is harmonic.

To satisfy the Cauchy-Riemann equations we want to solve for V from

$$U_x = V_y = 2y \qquad \text{and} \qquad -U_y = V_x = -2x.$$

So now we can solve this system of differential equations using integration:

$$\int V_y dy = \int 2y dy$$
$$V = y^2 + C_1(x)$$

where $C_1(x)$ is a constant in terms of y (but NOT necessarily in terms of x). Similarly

$$\int V_y dy = \int -2x dx$$
$$V = -x^2 + C_2(y)$$

where $C_2(y)$ is a constant in terms of x.

putting these together we have

$$V = y^2 + C_1(x) = -x^2 + C_2(y).$$

Therefore V must equal

$$V = y^2 - x^2 + C.$$

Example 9.13. *Find a harmonic conjugate of $U = y^3 - 3x^2y$.*

Taking partial derivatives we have

$$U_x = -6xy$$
$$U_y = 3y^2 - 3x^2$$
$$U_{xx} = -6y$$
$$U_{yy} = 6y.$$

Notice that $U_{xx} - U_{yy} = 0$, so U is harmonic.

Now since $V_y = U_x$ and $-V_x = U_y$ we have a differential equation. Using integration we get

$$\int V_y dy = \int -6xy dy$$
$$V = -3xy^2 + C_1(x),$$

and

$$\int V_x dx = \int -3y^2 + 3x^2 dx$$
$$V = -3xy^2 + x^3 + C_2(y).$$

So,

$$V = -3xy^2 + C_1(x) = -3xy^2 + x^3 + C_2(y).$$

Therefore

$$V = -3xy^2 + x^3 + C.$$

Example 9.14. *Find the harmonic conjugate of* $U = e^{-2x}\sin(2y)$.

We take partial derivatives:

$$V_y = U_x = -2e^{-2x}\sin(2y)$$
$$-V_x = U_y = 2e^{-2x}\cos(2y).$$

By integrating the partials of V *we get*

$$\int V_y dy = \int -2e^{-2x}\sin(2y)dy$$
$$V = e^{-2x}\cos(2y) + C_1(x),$$

and

$$\int V_x dx = \int -2e^{-2x}\cos(2y)dy$$
$$V = e^{-2x}\cos(2y) + C_2(y).$$

Now notice that

$$V = e^{-2x}\cos(2y) + C_1(x = e^{-2x}\cos(2y) + C_2(y)$$

so

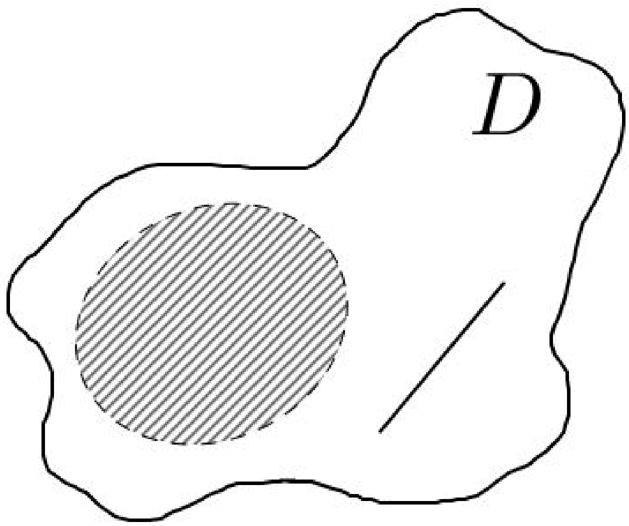
$$V = e^{-2x}\cos(2y) + C$$

10. Lecture 10: Uniqueness, Functions, and Branches

Uniqueness

Theorem 10.1. *Suppose f,g are analytic in domain D. Suppose further that $f = g$ at each point in an open set or a line segment in D. Then $f = g$ everywhere in D.*

The following is an illustration of a domain containing a line segment and an open set.



Later we will prove Theorem 10.1 with Taylor Series. Interestingly, we will see that the Taylor Series converges to the function everywhere in D is the function is analytic.

Recall that the Taylor series is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

We have the following beautiful theorem:

Theorem 10.2. *(Reflection Principle) Suppose D is a domain symmetric across the real axis. (In particular note that D must intersect the real axis since it is connected). Then we will have*

$$\overline{f(z)} = f(\overline{z}) \Leftrightarrow f \text{ is real when } y = 0$$
$$(z \in \mathbb{R} \to f(z) \in \mathbb{R}).$$

Proof. First we show that assuming $f(x+i0)$ is real implies $\overline{f(z)} = f(\overline{z})$. Notice that

$$f(x+i0) = u(x,0) = iv(x,0) = u(x,0).$$

Now let $g(z) = \overline{f(\overline{z})}$ and see that

$$\begin{aligned}\overline{f(\overline{z})} &= \overline{f(x-iy)} \\ &= \overline{u(x,-y)+iv(x,-y)} \\ &= u(x,-y) - iv(x,-y).\end{aligned}$$

Now define $\tilde{u} = u(x,-y)$ and $\tilde{v} = v(x,-y)$ so that $g(x+iy) = \tilde{u} - i\tilde{v}$. From this we get

$$\begin{aligned}\tilde{u} &= u(x,-y) & \tilde{v} &= -v(x,-y) \\ \tilde{U}_x &= U_x(x,-y) & \tilde{V}_x &= -V_x(x,-y) \\ \tilde{U}_y &= U_y(x,-y)(-1) & \tilde{V}_y &= V_y(x,-y) \\ &= V_x(x,-y). & &\end{aligned}$$

Notice $\tilde{U}_x = \tilde{V}_y$ and $\tilde{V}_x = -\tilde{U}_y$. So $g(z)$ is analytic.

Lets check $g(z)$ on the real line ($z = i0$)

$$f(x+i0) = u(x,-0) - iv(x,-0) \qquad = u(x,0) - iv(x,0),$$

by our starting assumption $u(x,0) = f(z+i0)$. So, $g(z) = f(z)$ on the real axis (a line segment) and by uniqueness this implies $g(z) = f(z)$ everywhere in D.

Thus, $g(z) = \overline{f(\overline{z})}$, which implies

$$\overline{\overline{f(\overline{z})}} = f(\overline{z}) = \overline{f(z)}.$$

This completes this direction of the proof.

Now we will show that by assuming $\overline{f(z)} = f(\overline{z})$ and $\overline{z} = x - iy$ it follows that $f(x+i0)$ is real. We have

$$\begin{aligned}\overline{u(x,y)+iv(x,y)} &= u(x,-y)+iv(x,-y) \\ u(x,y) - iv(x,y) &= u(x,-y)+iv(x,-y).\end{aligned}$$

Proceed by plugging in $y = 0$:

$$u(x,0) - iv(x,0) = u(x,0) + iv(x,0).$$

Putting all terms on the same side gives $0 = 2iv(x,0)$. Therefore $v(x,0)$ must equal 0. This means that $f(x+i0) = u(x,y) + i0$, a real function. □

Example 10.3. *Can we apply the reflection principal to $f(z) = z^2 - 2z$?*

Notice that $f(\overline{z}) = \overline{z}^2 - 2\overline{z}$. And

$$\begin{aligned}\overline{f(z)} &= \overline{z^2 - 2z} \\ &= \overline{z}^2 - 2\overline{z}.\end{aligned}$$

So $\overline{f(z)} = f(\overline{z})$. Moreover we have

$$x \in \mathbb{R} \to f(x) = x^2 - 2x \in \mathbb{R}.$$

Thus both assertions of the reflective principle are true.

Example 10.4. *Can we apply the reflection principal to $g(z) = z^2 - iz$?*

Notice that $g(\overline{z}) = \overline{z}^2 - i\overline{z}$. And

$$\begin{aligned}\overline{g(z)} &= \overline{z^2 - iz} \\ &= \overline{z}^2 + i\overline{z}.\end{aligned}$$

So $\overline{f(z)} \neq f(\overline{z})$. Moreover we have

$$x \in \mathbb{R} \to g(x) = x^2 - ix \notin \mathbb{R}.$$

Thus both assertions of the reflective principle are false.

Functions

We know $e^z = e^x \cos y + ie^x \sin y$.

Note: Normally $z^{1/n}$ is thot of as nth roots, but for exponential functions we will have

$$e^{1/n} = e^{1/n} \cos 0 + ie^{1/n} \sin 0,$$

which is just a the function $e^{1/n}$ on the real line.

Properties of Exponential Functions:

$$\begin{aligned}&e^{a+b} = e^a e^b \\ &e^{a-b} = e^a / e^b \\ &e^0 = 1 \\ &\frac{d}{dz} d^z = d^z \\ &e^z \neq 0 \text{ for all } z \in \mathbb{C}.\end{aligned}$$

These are all properties familiar to the real exponential function, but we also have some additional properties:

$$\begin{aligned}&|e^z| = |e^x||e^{iy}| = e^x \\ &e^{z+i2\pi k} = e^z.\end{aligned}$$

Notice that e^z is periodic in teh imaginary direction. Also, unlike in the real case, e^z may take on negative values; for example $e^{i\pi} = -1$.

Example 10.5. *Solve $e^z = 1 + i$ for z.*

Notice that

$$\begin{aligned}e^z &= 1 + i \\ e^x e^{iy} &= \sqrt{2} e^{i(\pi/4 + 2\pi k)}.\end{aligned}$$

Thus, $e^x = \sqrt{2} \to x = \ln \sqrt{2}$ and $y = \pi/4 + 2\pi k$.

Therefore $z = x + iy = \ln\sqrt{2} + i(\pi/2 + 2\pi k)$.
On a side note, this means that $\log(1+i) = \ln\sqrt{2} + i(\pi/4 + 2\pi k)$.

We can define the complex log more gerneraly. Since log is the inverse of e^z notice

$$e^{u+iv} = x + iy = re^{i\theta}.$$

So $e^u = r \to u = \ln r$ and $v = \theta + 2\pi k$. This means that

$$\log(z) = ln(r) + i(\theta + 2\pi k).$$

Example 10.6. *Find* $\log(-1)$. *This is just a direct computation:*

$$\begin{aligned}\log(-1) &= \log(e^{i\theta}) \\ &= \ln 1 + i(\pi + 2\pi k) \\ &= i\pi(1 + 2k).\end{aligned}$$

Branch

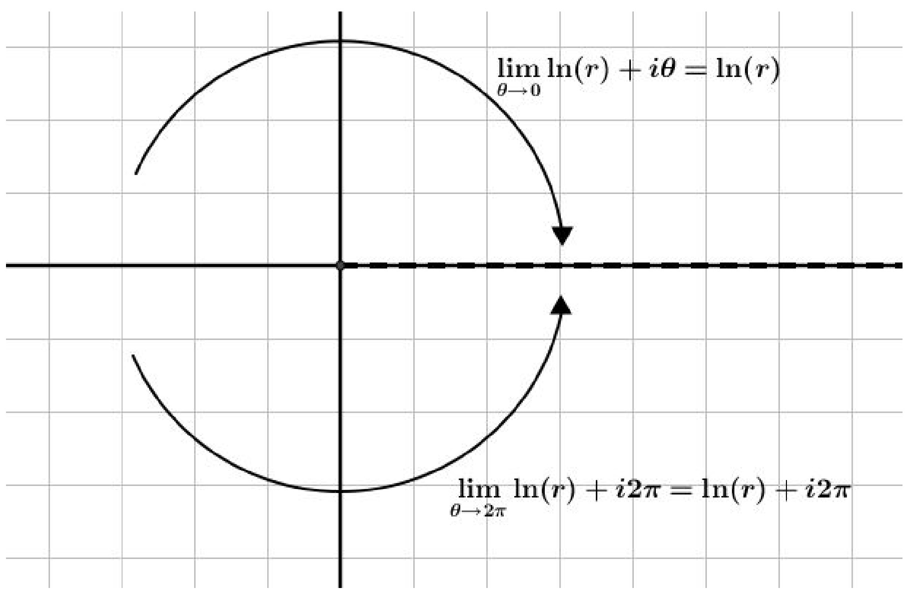
Definition 10.7. A **branch** is when we limit our angles θ to a 2π range

$$\alpha < \theta < \alpha + 2\pi$$

where we call α the branch cut.

If $-\pi < \theta < \pi$, (the principal branch) we will use $\text{Log}(z)$, (with a capitol 'L'). Note that $\log(z)$ is undefined at $\theta = \alpha$ (where α is the branch cut); $\log(z)$ jumps by $i2\pi$ across the branch cut.

For example if we have the branch $0 < \theta < 2\pi$, then $\log(z)$ will NOT be defined when $z = 0$ or when $\text{Arg}(z) = 0$.



Log Rules:

$$\log(ab) = \log(a) + \log(b)$$
$$\log(a/b) = \log(a) - \log(b)$$
$$\log(a^r) - r\log(a)$$

These always work for the multi-valued version of log, but NOT necessarily when we have a branch cut. Also, if we have a branch, then we will have the familiar derivative

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

Example 10.8. *Find Log*$(i(-2+2i))$ *and* $\mathrm{Log} i + \mathrm{Log}(-2+2i)$.

This can be done with log properties

$$\begin{aligned} Log(i(-2+2i)) &= Log(i(-2-2i)) \\ &= Log(\sqrt{8}e^{-i\frac{3\pi}{4}}) \\ &= \ln\sqrt{8} - i\frac{3\pi}{4}. \end{aligned}$$

Now we find $\mathrm{Log} i + \mathrm{Log}(-2+2i)$. *This is again an application of log properties.*

$$\begin{aligned} \mathrm{Log}(i) + \mathrm{Log}(-2+2i) &= \mathrm{Log}(e^{i\frac{\pi}{2}}) + \mathrm{Log}(\sqrt{8}e^{-i\frac{3\pi}{4}}) \\ &= \ln(1) + i\frac{\pi}{2} + \ln(\sqrt{8}) + i\frac{\pi}{3} \\ &= \ln(\sqrt{8}) + i\frac{5\pi}{4}. \end{aligned}$$

From these two computations we see that

$$\mathrm{Log}(i(-2+2i)) \neq \mathrm{Log}(i) + \mathrm{Log}(-2+2i).$$

Thus in general $\mathrm{Log}(zw) \neq \mathrm{Log}(z) + \mathrm{Log}(w)$ *when we have a branch.*

11. Lecture 11:Complex Powers, Trig Functions, Integrals, and Contours

Complex Powers

Definition 11.1.
$$z^c := e^{c\log(z)}.$$

Example 11.2. *Evaluate i^i.*
From our definition we have
$$\begin{aligned} i^i &= e^{i\log(i)} \\ &= e^{i(\ln(1)+(\frac{pi}{2}+2\pi k))} \\ &= e^{-(\frac{\pi}{2}+2\pi k)}. \end{aligned}$$
Interestingly we have that $i^i = e^{-(\frac{\pi}{2}+2\pi k)}$ is always a real number.

Normal rules for exponents still work,

$$\begin{aligned} z^{a+b} &= z^a z^b \\ z^{a-b} &= \frac{z^a}{z^b} \\ (z^a)^b &= z^{ab}. \end{aligned}$$

Additionally we have the familiar rules for z^c and c^z for some fixed $c \in \mathbb{C}$ differentiation,

$$\begin{aligned} \frac{d}{dz}z^c &= \frac{d}{dz}e^{c\log(z)}) \\ &= e^{c\log(z)} \cdot \frac{c}{z} \\ &= z^c \cdot \frac{c}{z} \\ &= cz^{c-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dz}c^z &= \frac{d}{dz}e^{z\log(c)} \\ &= e^{z\log(c)}\log(c) \\ &= c^z\log(c). \end{aligned}$$

Examples 11.3.

$$\frac{d}{dz}2^x \qquad = 2^x \ln(2)$$
$$\frac{d}{dz}i^z = i^z \log(i).$$

Trig

Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. From this we can obtain the following formula for $\cos(\theta)$

$$\begin{aligned} e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \\ e^{i\theta} + e^{-i\theta} &= 2\cos(\theta) \\ \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \end{aligned}$$

and a similar formula for $\sin(\theta)$,

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2i\sin(\theta) \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

These should still work for complex θ, so we get the complex definition of the trig functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$

from which all other trig functions follow.

Properties of Trig Functions:

$$\begin{aligned} &\frac{d}{dz}\sin(z) = \cos(z) \\ &\frac{d}{dz}\cos(z) = -\sin(z) \\ &\sin^2 z + \cos^2 z = 1 \\ &\sin(z + 2\pi) = \sin(z) \\ &\cos(z + 2\pi) = \cos(z). \end{aligned}$$

Also $\sin(z)$ is and odd function and $\cos(z)$ is an even function. All these properties are the same as they are in the real case. We show why $\sin^2 z + \cos^2 z = 1$:

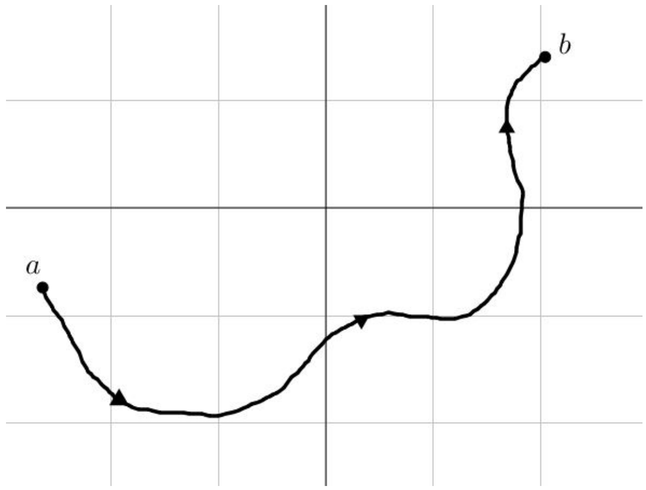
$$\begin{aligned} \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 &= \frac{(-e^{i2z} - e^{-i2z} + 2e^0) + (e^{i2z} + e^{-i2z} + 2e^0)}{4} \\ &= \frac{4}{4} \\ &= 1. \end{aligned}$$

Integrals

When we take integrals we will take them over **paths**. A parametrization of a path is

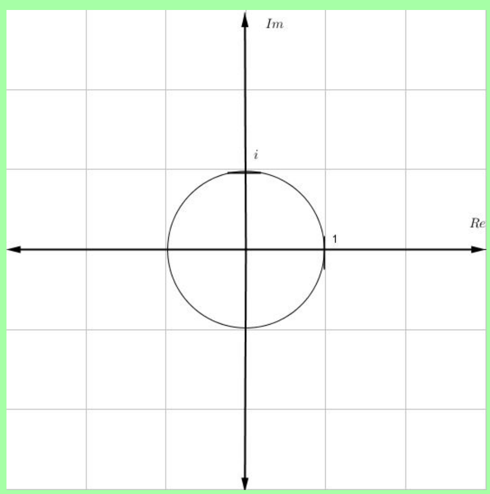
$$z(t) = x(t) + iy(t).$$

For example, the following diagram illustrated a path from a to b:



Example 11.4. *A familiar path is the unit circle:*

$$\cos\theta + i\sin\theta = e^{i\theta} = 1 \qquad \textit{and} \qquad 0 \leq \theta < 2\pi.$$



We can find the velocity, $z'(t)$ along a path s as follows,

$$z'(t) = x'(t) + iy'(t).$$

Taking the modulus of the velocity we obtain the speed,

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{x'(t)^2 + y'(t)^2}.$$

We can also take integrals of along paths. Given a path $w(t) = x(t) + iy(t)$,

$$\int_a^b w(t)dt = \int_a^b \mathrm{Re}(w(t))dt + i\int_a^b \mathrm{Im}(w(t))dt.$$

Example 11.5. *Find the integral along the path $w(t) = (1+it)^2$ from* 0 *to* 1.

We can do this with as we described above;

$$\begin{aligned}\int_0^1 (1+it)^2 dt &= \int_0^1 1 - t^2 dt + i\int_0^1 2t dt \\ &= t - \frac{t^3}{3}\Big|_0^1 + i\left(t^2\Big|_0^1\right) \\ &= 1 - \frac{1}{3} + i \\ &= \frac{2}{3} + i.\end{aligned}$$

Lets see what happens if, instead of splitting the integral into real and imaginary part, we just integrate directly from $w(t)$:

$$\begin{aligned}\int_0^1 (1+it)^2 dt &= \int_0^1 \frac{(1+it)^2}{i} d(1+it) \\ &= \frac{(1+it)^3}{3i}\Big|_0^1 \\ &= \frac{1}{3i}((1+i)^3 - 1^3) \\ &= \frac{2}{3} + i.\end{aligned}$$

See that we get the same result this way.

Example 11.6. *Calculate the integral $\int_0^{\pi/4} e^{i\theta} d\theta$.*

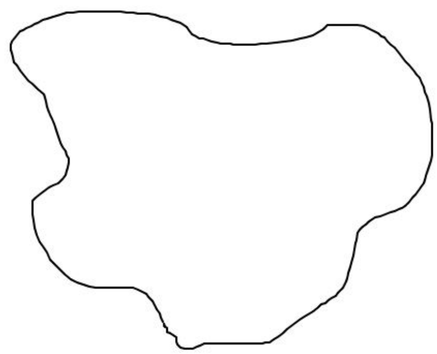
$$\begin{aligned}\int_0^{\pi/4} e^{i\theta} d\theta &= \frac{e^{i\theta}}{i}\Big|_0^{\pi/4} \\ &= \frac{e^{i\pi/4}}{i} - \frac{e^0}{i} \\ &= \frac{\sqrt{2}}{2} - i\left(1 - \frac{\sqrt{2}}{2}\right).\end{aligned}$$

Contours

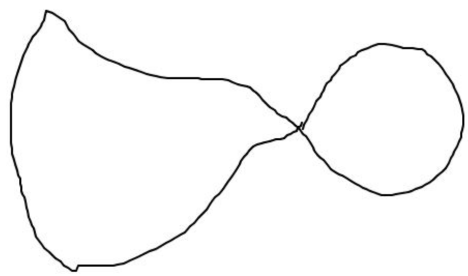
Definition 11.7. A **contour** (also known as a **curve**) is a path in the complex plain thot of as a shape (as opposed to a parametrization).

Definition 11.8. A contour is **simple** if it does not cross itself.

Definition 11.9. A **closed** contour is one that starts and ends at the same point.



Simple Closed Countour

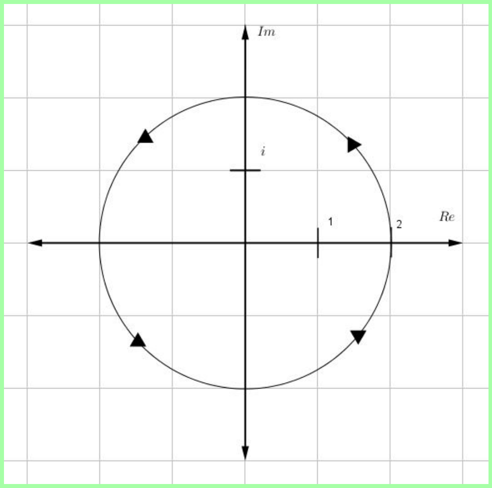


Non-Simple Closed Contour

For simple closed contours we consider counterclockwise to be the positive direction, or positive orientation.

12. Lecture 12:Integrals over paths

Example 12.1. *Calculate* $\int_C \frac{1}{z} dz$ *where* C *is the following simple closed positively oriented contour:*



Firstly we parametrize C *as*

$$z = 2e^{i\theta} \qquad \text{for} \qquad 0 \leq \theta \leq 2\pi.$$

From this we also have that $dz = 2ie^{i\theta} d\theta$*. Thus,*

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{2e^{i\theta}} d\theta \\ &= \int_0^{2\pi} i d\theta \\ &= 2\pi i.\end{aligned}$$

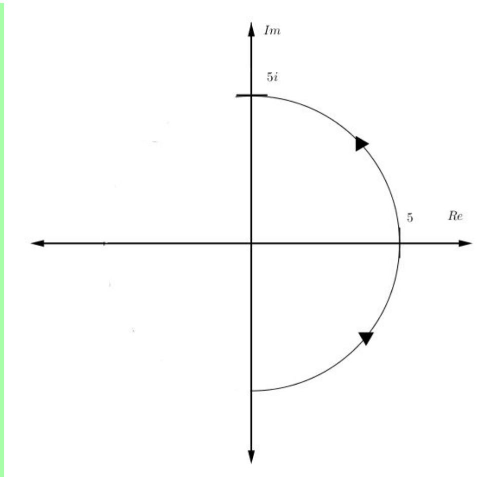
In general for a given parametrization of a contour C, $z(t)$ for $a \leq t \leq b$, we can find the integral of $f(z)$ over C as

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

Remark 12.2. The following integral gives the length of the curve C:

$$\int_a^b |z'(t)| dt.$$

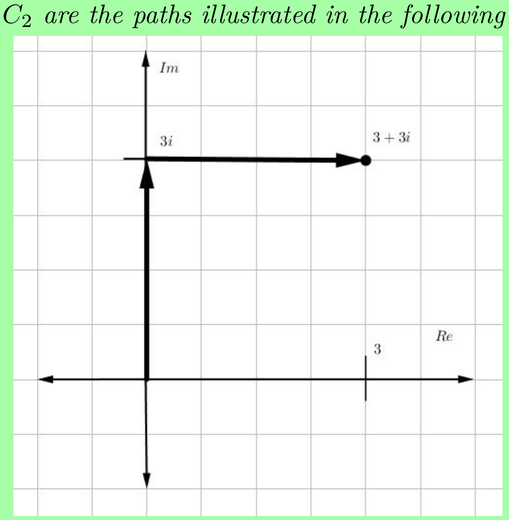
Example 12.3. *If* C *is the right half-circle of radius 5 centered at* 0 *and oriented counter clockwise (as illustrated in the diagram) then compute* $\int_C \overline{z} dz$*.*



A parametrization of C is $z(\theta) = 5e^{i\theta}$ for $-\pi/2 \leq \theta \leq \pi/2$. From this we also have that $dz = 5ie^{i\theta}d\theta$. Thus

$$\begin{aligned}\int_C \overline{z} &= \int_{-\pi/2}^{\pi/2} \overline{(5e^{i\theta})} i5e^{i\theta} d\theta \\ &= \int_{-\pi/2}^{\pi/2} 5e^{-i\theta} i5e^{i\theta} d\theta \\ &= \int_{-\pi/2}^{\pi/2} 25i d\theta \\ &= 25\pi i.\end{aligned}$$

Example 12.4. *Integrate $f(z) = y - x - i3x^2$ along the path $C = C_1 + C_2$, where C_1 and C_2 are the paths illustrated in the following diagram.*



Notice we can split up this integral as follows

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

We can also find parametrization for C_1 and C_2:

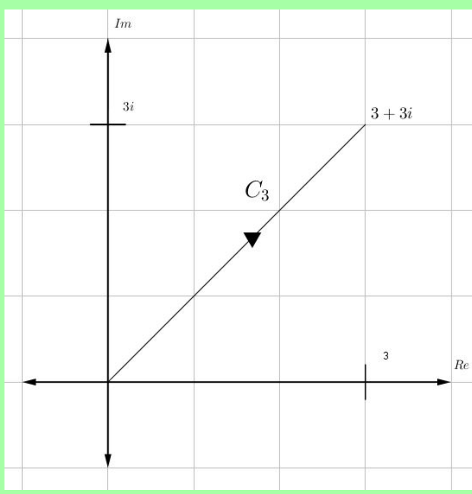
$$\begin{array}{lll} C_1: \; 3it & 0 \le t \le 1 & dz = 3idt \\ C_2: \; 3i + 3t & 0 \le t \le 1 & dz = 3dt \end{array}$$

Thus,

$$\begin{aligned} \int_C f(z)dz &= \int_0^1 (3t - 0 - i3(0)^2)3idt + \int_0^1 (3 - 3t - i3(3t)^2)3dt \\ &= (\frac{3}{2}t^2)(3i)\Big|_0^1 + (3t - \frac{3}{2}t^2 - i9t^3)3\Big|_0^1 \\ &= \frac{9i}{2} + (3 - \frac{3}{2} - i9)3 \\ &= \frac{9}{2} - i\frac{45}{2}. \end{aligned}$$

Now lets try a similar integral.

Example 12.5. *Integrate $f(z) = y - x - i3x^2$ over C_3.*



We can parametrize C_3 as

$$C_3: \; (3+3i)t \; 0 \le t \le 1 \qquad dz = (3+3i)dt.$$

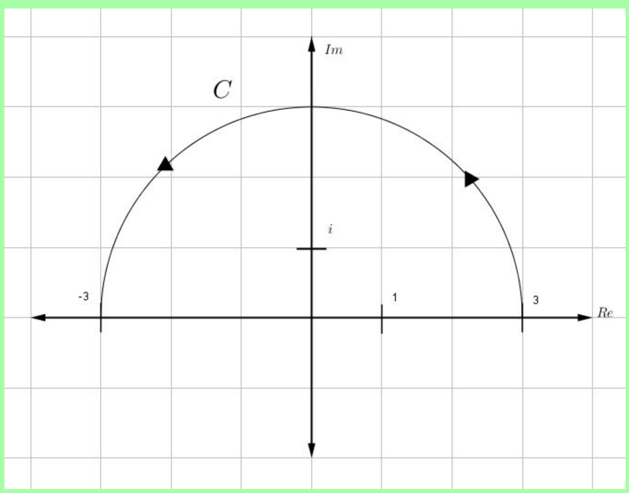
Now integrating we have

$$\begin{aligned}\int_{C_3} f(z)dz &= \int_0^1 (3t - 3t - i3(3t)^2)(3+3i)dt \\ &= (-i9t^3)(3+3i)\Big|_0^1 \\ &= -i27 + 27.\end{aligned}$$

Now notice that the previous two examples show that the integral depends upon the path we take, despite having the same starting and ending points. However, we will later see that if $f(z)$ is analytic we do in fact get the same solution of the integral regardless of the path we choose.

We close out this lecture with one final example.

Example 12.6. *integrate* $\int_C z^{1/2}dz$ *over the positively oriented upper half circle with radius* 3.



Since $z^{1/2}$ *is multivalued, we have to pick a branch. We will choose the branch* $-\pi < 0 < 3\pi/2$.

We can parametrize our path of integration as $z(\theta)$ for $0 \leq \theta \leq \pi$ and find that $dz = 3ie^{i\theta}d\theta$. And now we can compute the integral:

$$\begin{aligned}
\int_C z^{1/2} dz &= \int_0^{\pi} (3e^{i\theta})^{1/2} 3ie^{i\theta} d\theta \\
&= \int_0^{\pi} 3^{3/2} e^{i3\theta/2} i d\theta \\
&= 3^{3/2} \left(\frac{i}{i^{3/2}} \right) e^{i3\theta/2} \Big|_0^{\pi} \\
&= 3^{1/2} 2 e^{i3\theta/2} \Big|_0^{\pi} \\
&= 2\sqrt{3} \left(e^{i3\pi/2} - e^0 \right) \\
&= 2\sqrt{3}(-i - 1).
\end{aligned}$$

13. Lecture 13: Bounds on Integrals and Antiderivatives

Sometimes it is not necessary to compute exact values for an integral. For example if we can show

$$\left|\int_a^b z(t)dt\right| = r_0 e^{i\theta_0}.$$

For some complex number $r_0 e^{i\theta_0}$. From this we know

$$\left|\int_a^b z(t)dt\right| = \left|r_0 e^{i\theta_0}\right| = r_0.$$

But notice

$$\begin{aligned} r_0 &= e^{-i\theta_0}\int_a^b z(t)dt \\ &= \int_a^b e^{i\theta_0} z(t)dt. \end{aligned}$$

Note that r_0 is a positive real number and therefore the imaginary part of $\int_a^b e^{i\theta_0} z(t)dt$ must be zero. From this we have

$$r_0 = \int_a^b e^{i\theta} z(t)dt = \int_a^b \mathrm{Re}(e^{i\theta_0} z(t))dt.$$

Recall that $\mathrm{Re}(z) \le |z|$, so

$$r_0 = \int_a^b \mathrm{Re}(e^{i\theta_0} z(t))dt \le \int_a^b |e^{-i\theta}| \cdot |z(t)|dt = \int_a^b |z(t)|dt.$$

To summarize, we have shown the following:

Theorem 13.1 (Triangle Inequality for Integrals)**.**

$$\left|\int_a^b z(t)dt\right| \le \int_a^b |z(t)|\,dt.$$

Remark 13.2. Notice the similarity between this and the standard triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

We now prove the following useful theorem:

Theorem 13.3. *Assume C is a contour with length L and $|f(z)| \le M$ for all z on C. Then*

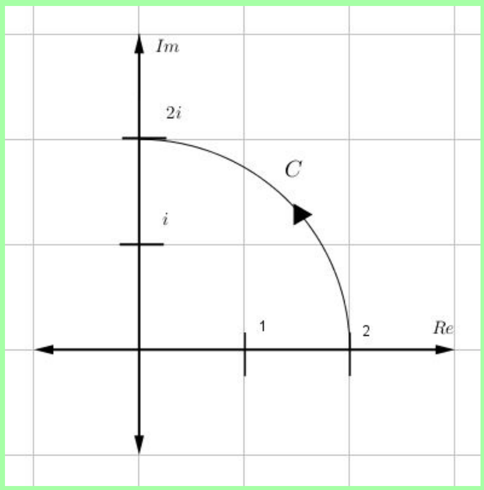
$$\left|\int_C f(z)dz\right| \le M \cdot L$$

Proof. Assume $z(t)$ for $a \le t \le b$ is a parametrization of C. Then we have

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \le \int_a^b |f(z(t))| \cdot |z'(t)|dt \le M \int_a^b |z'(t)|dt = ML.$$

For the last equality note that $\int_a^b |z'(t)|dt$ is the legnth of our contour C, which we denote L. □

Example 13.4. *Given the contour C illustrated in the following diagram:*



find a bound for $|\int_C \frac{z+4}{z^3-1}dz|$.

We have that

$$\left|\int_c \frac{z+4}{z^3-1}\right| \le M \cdot L$$

for some M and L which me must find. L is easy to find since it is just the length of our contour, $\frac{4\pi}{4} = \pi$.

Finding M is not quite as easy. First we notice that

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|}.$$

Using the triangle inequality we can show

$$|z+4| \le |z| + |4| = 2 + 4 = 6$$
$$|z^3-1| \ge ||z|^3 - |1|| = |2^3-1| = 7.$$

Notice that we have used $|z| = 2$, since this holds for all z in C. From these inequalities we have

$$\frac{|z+4|}{|z^3-1|} \le \frac{6}{7} = M.$$

So putting this altogether we have the bound

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}.$$

Example 13.5. *Find a bound for* $\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{1/2}}{z^2+1}\right|$*, where* C_R *is the origin centered half circle in the upper half plane with radius* R*.*
Additionally we choose the branch $-\pi/2<\theta<3\pi/2$*. We must choose a branch since* $z^{1/2}$ *is multivalued. We are trying to find* M *and* L *so that*

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{z^{1/2}}{z^2+1}\right|\leq ML.$$

It is easy to see that the length of our curve is just $L=\pi R$*.*

Notice that $|z|^{1/2}=R^{1/2}$*, this will be helpful in finding* M*. By the triangle inequality*

$$|z^2+1|\geq ||z^2|-1|=|R^2-1|=R^2-1$$

for R *sufficiently large. From this we can obtain our bound:*

$$\lim_{R\to\infty}\left|\int_C\frac{z^{1/2}}{z^2+1}dz\right|\leq\lim_{R\to\infty}\left(\frac{R^{1/2}}{R^2-1}\right)(\pi R)=0.$$

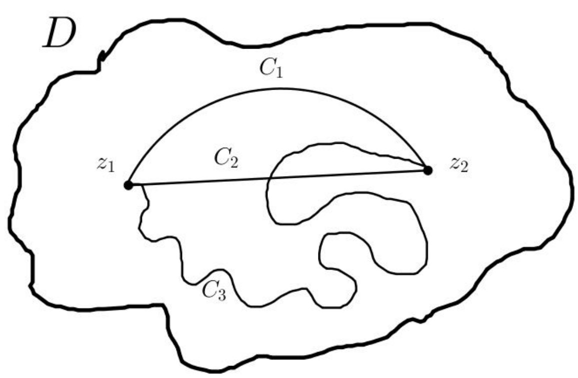
This means that

$$\lim_{R\to\infty}\left|\int_C\frac{z^{1/2}}{z^2+1}dz\right|=0$$

and

$$\lim_{R\to\infty}\int_C\frac{z^{1/2}}{z^2+1}dz=0.$$

13.1. **Antiderivatives.**



In the above diagram we have

$$\int_{C_1}f(z)dz=\int_{C_2}f(z)dz=\int_{C_3}f(z)dz=F(z_2)-F(z_1).$$

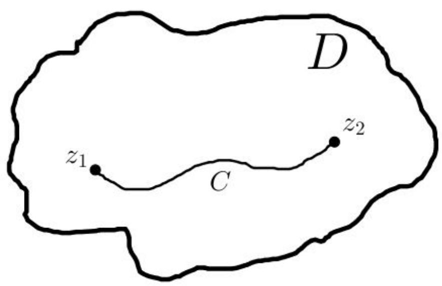
In particular, notice the similarity to the fundamental theorem of Calculus.

Theorem 13.6. *Suppose* $f(z)$ *is continuous on domain* D*. Then the following are equivalent:*

(1) *$f(z)$ has an antiderivative $F(z)$ defined thru out D. (Notice $F'(z) = f(z)$).*
(2) *For any 2 points $z_1, z_2 \in D$, all integrals of $f(z)$ on contours in D from z_1 to z_2 give the same value.*
(3) *Integrals of $f(z)$ around closed contours in D equal 0.*

Proof. (2) $\Rightarrow$ (3) is clear since to get a closed contour we just let the starting point z_1, and ending point z_2 coincide.

(1) $\Rightarrow$ (2). Assume we have $F(z)$ with $F'(z) = f(z)$.



Let $z_1, z_2 \in D$ and C be a contour from z_1 to z_2. Assume further that we can parametrize our contour as a function of t, $z(t)$ for $a \leq t \leq b$ where $z(a) = z_1$ and $z(b) = z_2$. From this parametrization we obtain the following real integral

$$\int_C f(z)dt = \int_a^b f(z(t))z'(t)dt.$$

But notice

$$\frac{d}{dt}(F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t).$$

Using this we have

$$\begin{aligned}\int_C f(z)dz &= \int_a^b \frac{d}{dt}\left(F(z(t))\right)dt \\ &= F(z(t))\Big|_a^b \\ &= F(z(b)) - F(z(a)) \\ &= F(z_2) - F(z_1).\end{aligned}$$

Notice that this final expression for our integral depends only upon the starting and ending points of the contour z_1 and z_2, and thus is path independent.

(2) $\Rightarrow$ (1). We need to find the antiderivative $F(z)$. To do this first pick some $z_0 \in D$.

We now define $F(z)$:

$$F(z) := \int_{z_0} zf(s)ds.$$

For some small Δz we have

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(s)ds$$

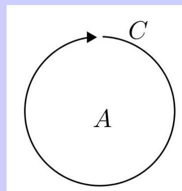
Notice we also have
$$\int_z^{z+\Delta z} f(z)ds = f(z)\Delta z.$$
This suggests that
$$\lim_{\Delta z \to 0} \left[\frac{F(z+\Delta z) - F(z)}{\Delta z} - f'(z) \right].$$
To show this notice
$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - \frac{f(z)\Delta z}{\Delta z} = \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} (f(s) - f(z))ds \right).$$
So, for any $\epsilon > 0$ we can find a δ so that $|s - z| \leq |\Delta z| < \delta$. Then $|f(s) - f(z)| < \epsilon$ and
$$\left| \frac{1}{\Delta z} \right| \cdot \left| \int_z^{z+\delta z} f(s) - f(z)ds \right| \leq \frac{1}{|\Delta z|}(\epsilon)|\Delta z| = \epsilon.$$
So we see that ϵ goes to 0 as $|\Delta z|$ goes to zero. □

14. Lecture 14:Integrals

We will need the following theorem from Calculus III.

Theorem 14.1 (Green's Theorem)**.**

$$\int_C Pdx + Qdy = \iint_A (Q_x - P_y)dA.$$



Theorem 14.2 (Cauchy-Goursat)**.** *Suppose $f(z)$ is analytic in domain D, and C is a simple closed contour in D and that everything inside C is contained in D. Then,*

$$\int_C f(z)dz = 0.$$

Proof. Let the parametrization of C be $z(t)$ where $a \leq t \leq b$. Let us write $z(t) = x(t) + iy(t)$. Then we have

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_a^b (u+iv)(x'+iy')dt \\ &= \int_a^b (u+iv)(dx+idy) \\ &= \int_C udx - vdy + i\int_C udy + vdx.\end{aligned}$$

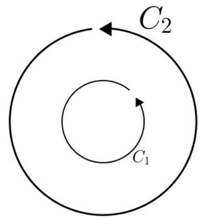
And by Green's theorem

$$\int_C f(z)dz = \iint_A -v_x - u_y dA + i\iint_A u_x - v_y dA.$$

But by the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, and so our integral becomes zero. □

Theorem 14.3 (Deform Paths)**.** *Let C_1 and C_2 be positively oriented, simple, closed contours with C_1 inside of C_2. If $f(z)$ is analytic between C_1 and C_2, then $\in_{C_1} f(z)dz = \int_{C_2} f(z)dz$.*

Proof. Consider the following diagram:



From this diagram,

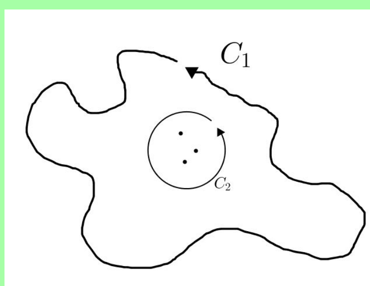
$$\int_C f(z)dz = \int_{C_2} f(z)dz + \int_{C_i} f(z)dz + \int_{C_o} f(z)dz - \int_{C_1} f(z)dz.$$

But we know that this sum of integrals equals zero by the Cauchy-Goursat Theorem. Notice however, that the contour $C_i = -C_o$ as the distance between C_i and C_o approaches zero. Thus we have

$$0 = \int_{C_2} f(z)dz - \int_{C_1} f(z)dz.$$

□

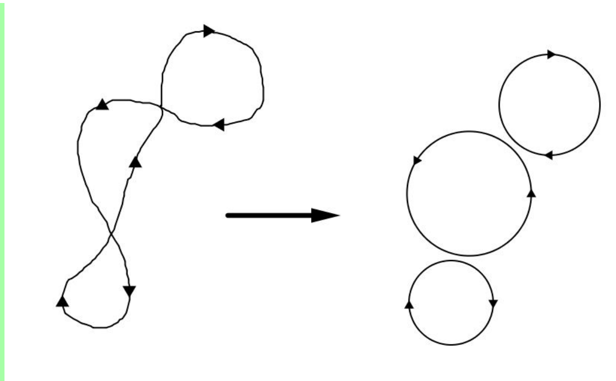
Example 14.4. *For example in the following diagram we can deform the contour C_1 to the contour C_2.*



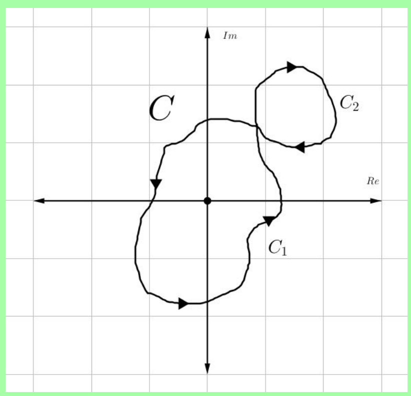
In this diagram we have denoted singularities with dots. Note that in our deformation process we cannot deform the contour 'thru' any singularity.

In the following example we deform a non-simple countour into several simple contours.

Example 14.5. *Notice that we can break an integral of a non-simple contour into separate parts where each part is a simple contour:*



Example 14.6. *Integrate $f(z) = 1/z$ over the contour C:*



Separating the non-simple contour C into the two simple contours C_1 and C_2 we have

$$\begin{aligned}\int_C \frac{1}{z}dz &= \int_{C_1} \frac{1}{z}dz + \int_{C_2} \frac{1}{z}dz\\ &= \int_{C_1} \frac{1}{z}dz.\end{aligned}$$

By deformation we can turn C_1 into C_r, where C_r is an origin centered circle with radius r small enough so that C_r is inside C_1. A parametrization for C_r is $z = re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, and from this we have $dz = rie^{i\theta}d\theta$. Using this we can compute the integral:

$$\begin{aligned}\int_{C_r} \frac{1}{z}dz &= \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta\\ &= \int_0^{2\pi} id\theta\\ &= 2\pi i.\end{aligned}$$

Theorem 14.7 (Cauchy's Integral Formula (CIF))**.** *Let $f(z)$ be analytic everywhere inside and on a contour C which is positively oriented, simple, and closed. If z_0 is inside C then,*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz.$$

(equivalently $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$)

Proof. Let $r > 0$ be small enough so that $|z - z_0| = r$ is inside C. Then

$$\int_C \frac{f(z)}{z - z_0}dz = \int_{C_r} \frac{f(z)}{z - z_0}dz.$$

Now consider

$$\begin{aligned}\int_C \frac{f(z)}{z - z_0}dz - 2\pi i f(z_0) &= \int_{C_r} \frac{f(z)}{z - z_0}dz - \int_{C_r} \frac{f(z_0)}{z - z_0}dz \\ &= \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}dz.\end{aligned}$$

Since $f(z)$ is continuous there is a δ so that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \frac{\epsilon}{2\pi}$. In fact we can come up with a bound:

$$\left|\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\right| \leq M \cdot L.$$

It is easy to find that the length of our curve L, is $2\pi r$. We can also find that M, the maximum value of $\left|\frac{f(z)-f(z_0)}{z-z_0}\right|$ in our domain, is less than $\frac{\epsilon/(2\pi)}{r}$. Thus our bound becomes

$$\left|\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\right| < \epsilon$$

Moreover, for any $\epsilon > 0$ we have

$$\left|\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} - 2\pi i f(z_0)\right| < \epsilon.$$

This implies

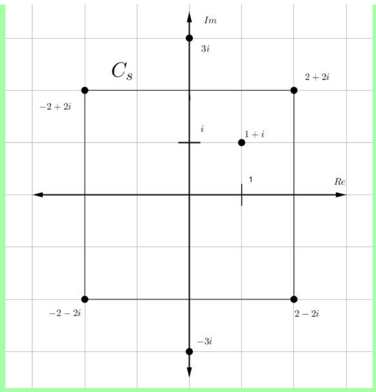
$$\left|\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} - 2\pi i f(z_0)\right| = 0$$

and thus

$$\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} = 2\pi i f(z_0).$$

□

Examples 14.8. *Consider the following contour C_s:*



- *Compute* $\int_{C_s} \frac{z}{(z^2+9)(z-1-ui)} dz$.
  Notice that we have singularities at $z = 3i$, $z = -3i$, *and* $1+i$. *But only one of these,* $1+i$, *is in our contour so that is the only one we will need to worry about for our integral. We now rewrite our integral and apply Cauchy's Integral Formula:*

$$\begin{aligned}\int_{C_s} \frac{z}{(z^2+9)(z-1-ui)} dz &= \int_{C_s} \frac{\left(\frac{z}{z^2+9}\right)}{z-(1+i)} dz \\ &= e\pi i f(1+i) \\ &= 2\pi i \frac{1+i}{(1+i)^2+9} = \frac{-14\pi}{85} + \frac{22\pi}{85} i.\end{aligned}$$

- *Compute the integral* $\int_{C_s} \frac{e^z}{z^2-25} dz$.
  This one is easy. First notice that the singularities are ± 5. *But both of these are outside of our contour and thus our integral just becomes* 0.

- *Compute* $\int_{C_s} \frac{\cos(z)}{z^2+4z} dz$.
  Firstly we notice the singularities are at 0 *and* -4. *Out of these two only* 0 *is inside the contour. Rewriting the integral and applying Cauchy's Formula we get,*

$$\begin{aligned}\int_{C_s} \frac{\cos(z)}{z^2+4z} dz &= \int_{C_s} \frac{\left(\frac{\cos(z)}{z+4}\right)}{z} dz \\ &= 2\pi i f(0) \\ &= 2\pi i \frac{\cos(0)}{4} \\ &= \frac{\pi i}{2}.\end{aligned}$$

Lets see what happens when we play around with Cauchy's Formula by taking derivatives:

$$f(s) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-s}dz$$
$$f'(s) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-s)^2}dz$$
$$f''(s) = \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-s)^3}dz$$
$$\vdots \quad \vdots$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-s)^{n+1}}dz$$

What we have just observed is the extension of Cauchy's Integral Formula.

Theorem 14.9 (Cauchy's Integral Formula Extension)**.**

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0).$$

Example 14.10. *Compute* $\int_C \frac{e^{2z}}{z^4}dz$*, where* C *is the unit circle.*

Fist notice that we have the single singularity $z = 0$*, which is in* C*. By computing the first few deriviatives of* $f(z) = e^2z$ *we will be able to apply the extension of Cauchy's Formula:*

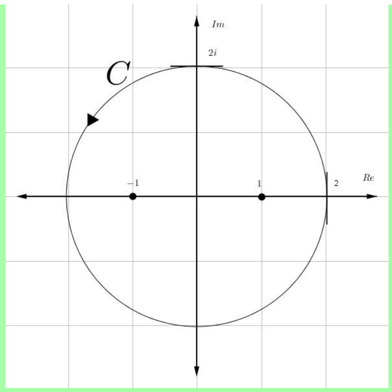
$$f'(z) = 2e^{2z}$$
$$f''(z) = 4e^{2z}$$
$$f'''(z) = 8e^{2z}.$$

Now we have

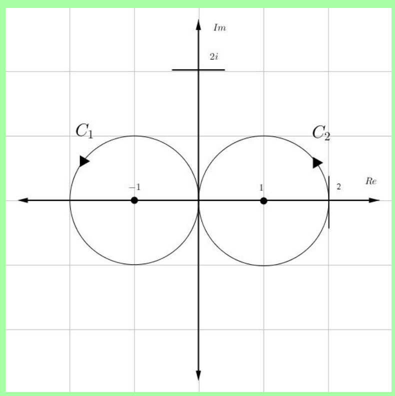
$$\int_C \frac{e^{2z}}{z^4}dz = \frac{2\pi i}{3!}\cdot 8e^0 = \frac{8\pi i}{3}.$$

Let us now do an example with two singularities inside our contour of integration.

Example 14.11. *Find the integral* $\int_C \frac{\cos(z)}{z^2-1}dz$*, where* z *is the origin centered circle of radius 2.*



First notice that we have two singularities, 1 *and* -1*, and that each of these are inside of* C*. By deforming* C *into* C_1 *and* C_2*, as shown in the following diagram, we turn our problem into two integrals which have single singularities inside of them (which we already know how to do).*



This makes the integral straightforward:

$$\begin{aligned}\int_C \frac{\cos(z)}{z^2-1}dz &= \int_{C_1} \frac{\cos(z)}{z^2-1}dz + \int_{C_2} \frac{\cos(z)}{z^2-1}dz \\ &= \int_{C_1} \frac{\left(\frac{\cos(z)}{z-1}\right)}{z+1}dz + \int_{C_2} \frac{\left(\frac{\cos(z)}{z+1}\right)}{z-1}dz \\ &= 2\pi i \frac{\cos(-1)}{(-1-1)} + 2\pi i \frac{\cos(1)}{1+1} \\ &= 0.\end{aligned}$$

15. Lecture 15:

Theorem 15.1. *If $f(z)$ is analytic at z_0, then $f'(z)$ is analytic at z_0.*

Corollary 15.2. *If $f(z)$ is analytic at z_0 it is infinitely differentiable at z_0.*

Remark 15.3. Earlier we used this fact (without proof) in our discussion of harmonic functions.

Proof. If $f(z)$ is analytic at z_0, then $f(z)$ is analytic in a neighborhood of z_0. Let

$$C: \quad |z - z_0| = r$$

be inside the neighborhood. From this and by Cauchy's Integral Formula we have

$$f''(z) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z)^3} dz.$$

Notice that this formula is true for any z inside C, so this implies that $f'(z)$ is analytic. □

Theorem 15.4 (Cauchy's Inequality)**.** *Suppose $f(z)$ is analytic inside and on a positively oriented circle, C_R, centered at z_0 and with radius R. Also let $|M_R| = \max|f(z)|$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$

for a positive integer n.

Proof. By the extension of Cauchy's Integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Taking the modulus of each side and bounding,

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2pii} \right| \cdot \left| \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} ML.$$

We find the length of the curve, $L = 2\pi R$.

Finding M:

$$\frac{|f(z)|}{|z-z_0|^{n+1}} \leq \frac{M_R}{R^{n+1}}.$$

Thus,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} ML \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} 2\pi R = n! \frac{M_R}{R^n}$$

□

Theorem 15.5 (Liouville's Theorem)**.** *If a function $f(z)$ is entire and bounded for all complex numbers. Then $f(z)$ is constant.*

Proof. By our assumption $f(z)$ is bounded and thus $|f(z)| \leq M$ for some M. Applying Cauchy's Inequality on a circle C_R, centered at z_0 and having radius R,

$$|f'(z)| \leq \frac{1! \cdot M}{R^1} = \frac{M}{R}.$$

This holds for all R, so lets take $R \to \infty$:

$$|f'(z)| \leq \frac{M}{\infty} = 0.$$

Thus $f'(z) = 0$ and $f(z)$ must be constant □

We now prove the familiar Fundamental Theorem of Algebra with complex analysis.

Theorem 15.6 (Fundamental Theorem of Algebra)**.** *Every polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, with degree $n \geq 1$ has a root, (a z_1 such that $P(z_1) = 0$). Then $P(z)$ can be factored $P(z) = (z - z_1)Q(z)$ where $Q(z)$ is an $(n-1)$th degree polynomial.*

Proof. By way of contradiction assume that $P(z)$ is a polynomial with degree $n \geq 1$, and $P(z) \neq 0$ for all $z \in \mathbb{C}$. Thus $f(z) = 1/P(z)$ should be entire. Expanding $P(z)$ we have

$$f(z) = \frac{1}{a_0 + a_1 z + \cdots + a_n z^n}.$$

So,

$$|p(z)| \geq ||a_n z^n| - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}||$$

for all $|z|$ sufficiently large, (say $|z| > R$). From this inequality we have

$$|a_n z^n| > 2|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|.$$

Thus

$$\big||a_n z^n| - |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|\big| > \frac{1}{2}|a_n z^n| > \frac{1}{2}|a_n| R^n.$$

So, $|f(z)| = \frac{1}{|P(z)|} < \frac{1}{\frac{1}{2}|a_n| R^n}$ for $|z| > R$.

Inside $|z| = R$ there are no asymptotes, so $|f(z)|$ is bounded inside the circle as well. Therefore, by Liouville's Theorem, $f(z)$is constant. But this contradicts the fact that $P(z)$ has degree $n \geq 1$. □

If z_1 is a root of an nth degree polynomial $P(z)$ $(P(z_1) = 0)$, then

$$\begin{aligned} P(z) &= P(z) - P(z_1) \\ &= a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \cdots + a_1(z - z_1) + a_0(1 - 1) \\ &= (z - z_1)Q(z), \end{aligned}$$

where $Q(z)$ is a polynomial with degree $n - 1$.

15.1. **Maximum Modulus Principle.**

Lemma 15.7. *If $f(z)$ is analytic in domain D, suppose $|f(z)|$ in D. Then $f(z)$ is constant in D.*

Proof. Notice $|f|^2 = f(z)\overline{f(z)} = C$, where C is some constant. From this we have $\overline{f(z)} = C/f(z)$. Thus $\overline{f(z)}$ is analytic. But now $f(z)$ and $\overline{f(z)}$ are analytic; recall tha this means $f(z)$ is constant. □

Lemma 15.8. *Suppose $|f(z)| \leq |f(z_0)|$ for all z in a neighborhood $|z - z_0| < \epsilon$. Then $f(z) = f(z_0)$ in the neighborhood.*

Proof. We choose $0 < r < \epsilon$ be the radius of a circle centered at z_0. Then by Cauchy's Integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z - z_0}dz.$$

Moreover, we have the following parametrization of C_r:

$$C_r : \ z(\theta) = z_0 + re^{i\theta} \qquad \text{for} \quad 0 \leq \theta \leq 2\pi.$$

And from this parametrization $dz = rie^{i\theta}d\theta$. Therefore

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0 + re^{i\theta})}{re^{i}\theta}ire^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

(This is known as Gauss's Mean Value Theorem).

From this we have

$$|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|d\theta = |f(z_0)|.$$

And thus

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + re^{i\theta})|d\theta.$$

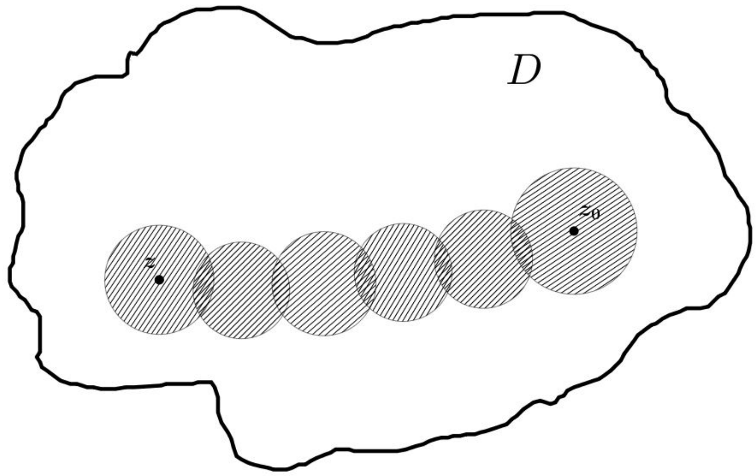
Since $|f(z_0)| \geq |f(z_0 + re^{i\theta})|d\theta$ we must actually have $|f(z_0)| = |f(z_0 + re^{i\theta}$. Noting that the 'biggest-spot' equals the average. So every spot must bee qual. In particualr

$$|f(z_0)| = |f(z)| \text{ everywhere in the neighorhood}$$

and $|f(z)|$ =constant $\Rightarrow f(z)$ is constant. □

Theorem 15.9 (Maximum Modulus Principle)**.** *If $f(z)$ is analytic in domain D, then either $f(z)$ is constant in D or $|f(z)|$ has no maximum in D (the maximum is either on boundary or at infinity).*

Proof. Suppose we have a max at z_0 in D. Then we know $f(z_0) \geq |f(z)|$ for all $z \in D$. But by the previous lemma we have that for z in a neighbor hood of z_0 we must have $f(z) = f(z_0)$. By covering D in overlapping neighborhoods we see that $f(z) = f(z_0)$ all for all $z \in D$.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□

16. Lecture 16: Series

16.1. **Sequences.** Sequences can be thot of as a 'list' of numbers:

$$z_1, z_2, z_3, \ldots$$

A sequence is said to have a limit at z_0 if, for any $\epsilon > 0$ we can find an N such that if $n \geq N$, then $|z - z_0| < \epsilon$. These means that every term past z_N is in a neighborhood of $|z - z_0| < \epsilon$.

Notation:

$$\lim_{n\to\infty} z_n = z_0.$$

A sequence with not limits is said to **diverge**.

Theorem 16.1. *If $z_n = x_n + iy_n$, has the limit $z_0 = x_0 + iy_0$, then*

$$\lim_{n\to\infty} z_n = z_0 \Leftrightarrow \lim_{n\to\infty} x_n = x_0 \text{ and } \lim_{n\to\infty} y_n = y_0.$$

16.2. **Series.** In general a series looks something like

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots = \lim_{n\to\infty} S_n$$

where

$$S_n = z_0 + z_1 + z_2 + \cdots + z_n.$$

If $\sum_{n=0}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$.

Remark 16.2. In Calculus II this was known as the 'Limit Test for Convergence.'

If $\sum_{n=0}^{\infty} |z_n|$ converges, then $\sum_{n=0}^{\infty} z_n|$ converges too. If $\sum_{n=0}^{\infty} |z_n|$ converges then we say the series is **Absolutely Convergent**.

Recall the familiar geometric series:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad \text{if } |z| < 1,$$

and diverges if $|z| \geq 1$.

16.3. **Taylor Series.** Some useful Taylor series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

In general the **Taylor Series** of a function is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

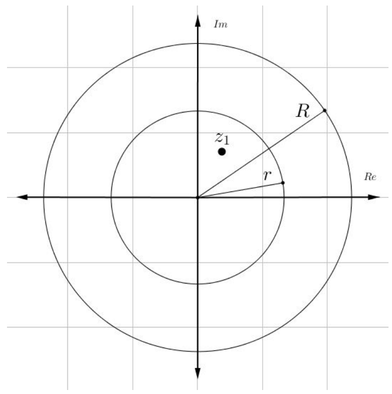
Theorem 16.3. *Suppose $f(z)$ is Analytic in a circle $|z - z_0| < R$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

inside the circle.

Proof. We will first prove for circles centered at zero, then generalize. For $z_0 = 0$ the Taylor series becomes

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$



Choose z, inside the circle. Pick r so that $|z_1| < r < R$. Then by Cauchy's Integral formula

$$f(z_1) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_1} dz.$$

But $\frac{1}{z - z_1} = \frac{1}{z} \cdot \frac{1}{1 - \frac{z_1}{z}}$, and since

$$|z_1| < |z| = r \Rightarrow \left| \frac{z_1}{z} \right| < 1,$$

we can apply the geometric series formula:

$$\frac{1}{z} \cdot \frac{1}{1 - \frac{z_1}{z}} = \frac{1}{z} \sum_{n=0"}^{\infty} \left(\frac{z_1}{z} \right)^n = \sum_{n=0}^{\infty} \frac{z_1^n}{z^{n+1}}.$$

Substituting this series into our formula for $f(z_1)$,

$$\begin{aligned} f(z_1) &= \frac{1}{2\pi i} \int_{C_r} f(z) \sum_{n=0}^{\infty} \frac{z_1^n}{z^{n+1}} dz \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{n+1}} dz \right) z_1^n. \end{aligned}$$

However, by the extension of Cauchy's Integral Formula

$$\int_{C_r} \frac{f(z)}{z^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(0).$$

Substituting this into our formula for $f(z_1)$

$$f(z_1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z_1^n.$$

This is the Taylor series centered at $z_0 = 0$ (aka the Maclaurin series).

To find the Taylor series centered at a general z_0 let $g(z) = f(z + z_0)$. We know $f(z)$ is analytic in $|z - z_0| < R$, which implies $g(z)$ is analytic in $|z| < R$. Explicitly,

$$\begin{aligned} f(z + z_0) = g(z) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n. \end{aligned}$$

And we have the Taylor series for general z_0,

$$f(z) = f(z - z_0 + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

□

Example 16.4. *Find the Taylor series of $f(z) = z^2 e^{3z}$ centered at 0.*

Notice $f(z)$ is entire, so the power series always works. We know

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ e^{3z} &= \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} \\ z^2 e^{3z} &= \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}. \end{aligned}$$

So

$$\begin{aligned} f(z) &= \frac{3^0 z^2}{0!} + \frac{3^1 z^3}{1!} + \frac{3^2 z^4}{2!} + \cdots \\ &= \sum_{n=2}^{\infty} \frac{3^{n-2} z^n}{(n-2)!}. \end{aligned}$$

This is our Taylor series.

Example 16.5. *Find the Taylor series of* $\sin(z)$ *centered at* $\pi/2$.

We know that the Taylor series should look something like

$$\sum_{n=0}^{\infty} (?)(z - \pi/2)^n.$$

From trigonometry we know $\sin(z) = \cos(z - \pi/2)$*. But we know the the Taylor series*

$$\cos(z - \pi/2) = \sum_{n=0}^{\infty} \frac{(z - \pi/2)^{2n}}{(2n)!}$$

which is also the Taylor series we want.

Example 16.6. *Find the Taylor Series for* $\frac{1}{z+4}$ *centered at* $z = 0$*.*

We will use the geometric series.

$$\begin{aligned} \frac{1}{z+4} &= \frac{1}{4(1 + \frac{z}{4})} \\ &= \frac{1}{4} \cdot \frac{1}{1 - (\frac{-z}{4})} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{-z}{4}\right)^n \end{aligned}$$

for $\left|\frac{-z}{4}\right| < 1$*. In other words*

$$\frac{1}{z+4} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{4^{n+1}} \qquad \text{for } |z| < 4.$$

Example 16.7. *Find the Taylor series of* $f(z) = \frac{1}{z+4}$ *centered at* $z = 2$*.*

Rewriting $f(z)$ *we have*

$$f(z) = \frac{1}{z+4} = \frac{1}{6 + (z-2)} = \frac{1}{6} \cdot \frac{1}{1 + \frac{z-2}{6}} = \frac{1}{6} \cdot \frac{1}{1 + \left(\frac{-(z-2)}{6}\right)}.$$

Applying the geometric series

$$f(z) = \frac{1}{6} \cdot \frac{1}{1 + \left(\frac{-(z-2)}{6}\right)} = \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{-(z-2)}{6}\right)^n$$

when $\left|\frac{-(z-2)}{6}\right| < 1$*. Simplifying this we obtain the Taylor series*

$$\frac{1}{z+4} = \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{6^{n+1}} \qquad \text{for } |z-2| < 6.$$

Remark 16.8. In the previous example we say that 6 is the *radius of convergence*.

Example 16.9. *Find the Taylor series of* $f(z) = \frac{1}{z}$ *centered at* $2i$*.*

First write $f(z)$ as

$$f(z) = \frac{1}{z} = \frac{1}{z - 2i + 2i} = \frac{1}{2i} \frac{1}{1 - \left(\frac{-(z-2i)}{2i}\right)}.$$

And applying the geometric series formula

$$\begin{aligned} f(z) &= \frac{1}{2i} \frac{1}{1 - \left(\frac{-(z-2i)}{2i}\right)} \\ &= \frac{1}{2i} \sum_{n=0}^{\infty} \left(\frac{-(z-2i)}{2i}\right)^n \end{aligned}$$

for $\left|\frac{-(z-2i)}{2i}\right| < 1$. *Simplifying this we have the Taylor series*

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-2i)}{(2i)^{n+1}} \qquad \text{for } |z - 2i| < 2.$$

Example 16.10. *Find the Taylor series for e^z at $-5i$.*

We can write

$$e^z = e^{z+5i-5i} = e^{z+5i} e^{-5i}.$$

Using the Maclaurin series we have

$$e^z = e^{z+5i} e^{-5i} = e^{z+5i} \sum_{n=0}^{\infty} \frac{(z+5i)^n}{n!}.$$

Simplifying this we have the Taylor Series

$$e^z = \sum_{n=0}^{\infty} \frac{e^{-5i}(z+5i)^n}{n!}$$

17. Lecture 17: Laurent Series

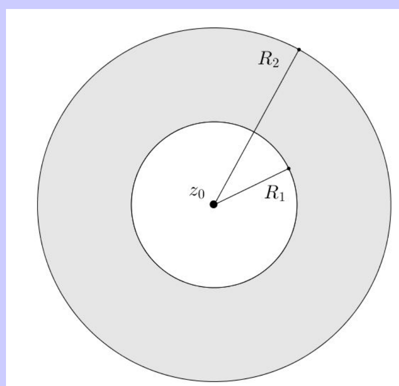
17.1. **Laurent Series.** For example $e^{1/z}$ is not, analytic at $z = 0$, so Taylor series doesn't work at $z = 0$.

But we do have the power series

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}.$$

This series works for $z \neq 0$. This is like Taylor series, but with negative powers also. We call such a series a Laurent series.

Theorem 17.1 (Existence of Laurent Series)**.** *Suppose $f(z)$ is analytic in the annulus $R_1 < |z - z_0| < R_2$.*



Let C denote a positively oriented simple closed contour around z_0 and lying completely in the annulus.

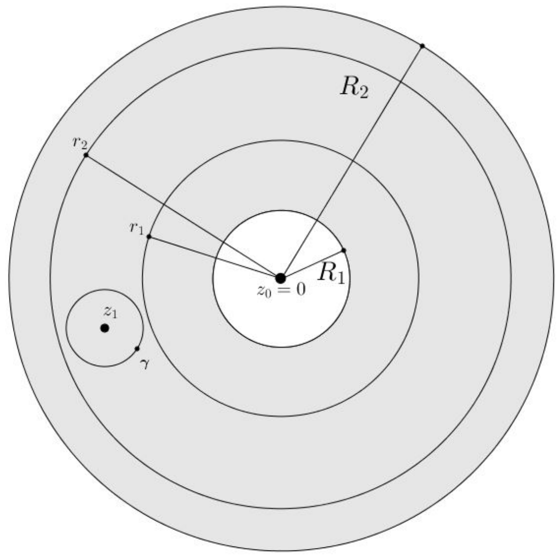
Then we have the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

on $R_1 < |z - z_0| < R_2$, and where

$$C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Proof. Start with $z_0 = 0$. Then fix z_1 in the annulus as illustrated in this diagram:



Choose r_1 and r_2 such that $R_1 < |z_1| < r_2 < R_2$. Let γ be a small circle around z_1. Then

$$\int_{C_{r_2}} \frac{f(z)}{z - z_1} dz - \int_\gamma \frac{f(z)}{z - z_1} dz - \int_{C_{r_1}} \frac{f(z)}{z - z_1} dz = 0$$

since (if you go along the paths in a certain way) the interior of the path of integration is analytic with no singularities we use Cauchy-Goursat. In particular we have

$$\int_\gamma \frac{f(z)}{z - z_1} dz = \int_{C_{r_2}} \frac{f(z)}{z - z_1} dz - \int_{C_{r_1}} \frac{f(z)}{z - z_1} dz.$$

But by Cauchy's integral formula,

$$\int_\gamma \frac{f(z)}{z - z_1} dz = 2\pi i f(z_1).$$

Thus

$$f(z_1) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)}{z - z_1} dz$$

On C_{r_2} $|z| > |z_1|$ so

$$\frac{1}{z - z_1} = \frac{1}{z} \frac{z}{z - \left(\frac{z_1}{z}\right)} = \frac{1}{z} \sum_{n=0}^\infty \left(\frac{z_1}{z}\right)^n = \sum_{n=0}^\infty \frac{z_1^n}{z^{n+1}}.$$

Conversely on C_{r_1} $|z| > |z_1|$ so,

$$\frac{-1}{z_1 - z} = \frac{1}{z_1} \frac{1}{1 - \frac{z}{z_1}} = \frac{1}{z_1} \sum_{n=0}^\infty \left(\frac{z}{z_1}\right)^n = \sum_{n=0}^\infty \frac{z^n}{z_1^{n+1}}$$

Rewriting this sum,

$$\sum_{n=0}^{\infty}\frac{z^n}{z_1^{n+1}} = \frac{1}{z_1}+\frac{z}{z_1^2}+\frac{z^2}{z_1^3}+\cdots$$
$$= \frac{z_1^{-1}}{1}+\frac{z_1^{-2}}{z^{-1}}+\frac{z_1^{-2}}{z^{-1}}+\frac{z_1^{-3}}{z^{-2}}+\cdots$$
$$= \sum_{n=-\infty}^{-1}\frac{z_1^n}{z^{n+1}}.$$

With these two series we can rewrite $f(z_1)$ as

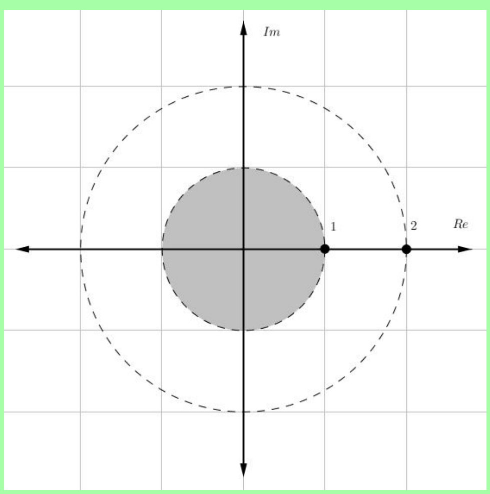
$$f(z_1) = \frac{1}{2\pi i}\int_{C_{r_1}} f(z)\sum_{n=0}^{\infty}\frac{z_1^n}{z^{n+1}}dz + \frac{1}{2\pi i}\int_{C_{r_2}} f(z)\sum_{n=-\infty}^{-1}\frac{z_1^n}{z^{n+1}}dz.$$

But we have that $C_{r_1} = C_{r_2}$ since our series are both analytic on our annulus, so,

$$f(z_1) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz\right))z_1^n.$$

And this is the Laurent series of $f(z)$ centered at $z_0 = 0$. For the Laurent series centered a t a general z_0 we can do a change of variables as we did in the Taylor series proof. □

Example 17.2. *Find the Laurent series of* $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ *centered at 0.*



We will end up with 3 series (due to locations of singularities). The three series will be for $|z| < 1$, $1 < |z| < 2$, *and* $|z| > 2$.

CASE 1 $(|z| < 1)$***:*** *For* $|z| < 1$ *we have*

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

for $|z| < 1$*. Similarly*

$$\frac{-1}{z-2} = \frac{1}{2-z} = \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{-z^n}{2^{n+1}}$$

for $|z| < 2$*. So combining these,*

$$\frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right) z^n$$

for $|z| < 1$*.*

CASE 2 $(1 < |z| < 2)$***:*** *For* $1 < |z| < 2$*,*

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{z - \frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

for $|z| > 1$*. Notice by expanding this sum we can rewrite it,*

$$\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$
$$= \sum_{n=-\infty}^{-1} z^n.$$

For $\frac{-1}{z-2}$ *we can use the sum from Case 1, since it was valid for* $|z| < 2$*. Thus for* $1 < |z| < 2$*.*

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n.$$

CASE 3 $(|z| > 2)$***:*** *For* $|z| > 2$ *we have*

$$\frac{1}{z-1} = \sum_{n=-\infty}^{-1} z^n.$$

Also

$$\frac{-1}{z-2} = \frac{-1}{z} \cdot \frac{1}{1-\frac{2}{z}} = \frac{-1}{z} \sum n = 0^{\infty} \left(\frac{2}{z}\right)^n.$$

Let us rewrite this sum

$$\frac{-1}{z} \sum n = 0^{\infty} \left(\frac{2}{z}\right)^n = \frac{-1}{z} + \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots$$
$$= -\sum_{n=-\infty}^{-1} \frac{z^n}{2^{n+1}}.$$

With these sums we find the following Laurent series for $|z| > 2$:

$$f(z) = \sum_{n=-\infty}^{-1} \left(1 - 2^{-n-1}\right) z^n$$

From our general Laurent series we have

$$C_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

And from this

$$C_{-1} = \frac{1}{2\pi i} \int_c f(z) dz,$$

which we can rewrite as

$$\int_c f(z) dz = 2\pi i C_{-1}.$$

17.2. **Properties of Taylor & Laurent series.** We give a list of some useful properties Taylor and Laurent series:

- Converges absolutely in a Circle/Annulus.
- We can add/subtract/derivative/antiderivative term by term.
- Unique for z_0, in an annulus (possibly a circle).

Note that tho we can divide and multiply these series, it is quite difficult in practice.

Now we will show two different ways to find the Laurent series of $\frac{1}{(1-z)^2}$. The first way is straightforward; just split the function into $\frac{1}{1-z} \cdot \frac{1}{1-z}$. In the second way we make the clever realization that $\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right)$.

Carrying out the 'fist-way' we have

$$\begin{aligned} \frac{1}{(1-z)^2} &= \frac{1}{1-z} \cdot \frac{1}{1-z} \\ &= \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{n=0}^{\infty} z^n\right) \\ &= (1 + z + z^2 + z^3 + \cdots)(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + 2z + 3z^2 + 4z^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1) z^n. \end{aligned}$$

The alternate way of deriving this Taylor series is by makeing the $\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right)$:

$$\begin{aligned}\frac{1}{(1-z)^2} &= \frac{d}{dz}\left(\frac{1}{1-z}\right) \\ &= \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \\ &= \sum_{n=0}^{\infty} nz^{n-1} \\ &= 0 + 1 + 2z + 3z^2 + 3z^2 + 4z^3 + \cdots\end{aligned}$$

Comparison of these two methods highlights the relative difficulty of multiplying power series as compared to differentiating power series.

18. Lecture 18: Series

Definition 18.1. A **singular point**, or **singularity**, of $f(z)$ is a point where $f(z)$ is not analytic but it is on the boundary of the set where $f(z)$ is analytic.

Definition 18.2. An **isolated singularity** is a point where $f(z)$ is not analytic, but $f(z)$ is analytic in the rest of a neighborhood.

Examples 18.3. :

- $\frac{1}{z}$ *has an isolated singularity at* $z = 0$.
- $\mathrm{Log}(z)$ *with branch* $-\pi < \theta < \pi$ *has singularities along the negative real axis, but no isolated singularities.*

Suppose $f(z)$ is analytic on the 'annulus' (in this case more of a 'punctured disk') $0 < |z - z_0| < R$. We know $f(z)$ has Laurent series

$$f(z) = \cdots + \frac{C_{-2}}{(z - z_0)^2} + \frac{C_{-1}}{(z - z_0)} + C_0 + C_1(z - z_0) + C_2(z - z_0)^2 + \cdots$$

Referring to this Laurent series we make the following definition.

Definition 18.4. The C_{-1} coefficient of the Laurent series of $f(z)$ (as written above) is called the **residue** of $f(z)$ at z_0 and is denoted $\operatorname*{Res}_{z_0}(f)$.

We have the formula

$$\operatorname*{Res}_{z_0}(f) = C_{-1} = \frac{1}{2\pi i} \int_C f(z) dz$$

where C is a simple, closed, positively oriented contour around z_0. We can rewrite the above equation in a useful way:

$$\int_C f(z) dz = 2\pi i \cdot \operatorname*{Res}_{z_0}(f).$$

Remark 18.5. One may wonder where the term '*residue*' comes from.

If we consider taking the integral of the Laurent series, we can apply Cauchy Goursat to most of the terms to find out that they disapear. The only term that remains after integration is $\frac{C_1}{z - z_0}$. In particular

$$\int \frac{C_{-1}}{z - z_0} dz = 2\pi i C_{-1}.$$

And we see that the *residue* is the only part left after integration.

Example 18.6. *Find* $\int_C z^2 \sin(1/z) dz$ *where* C *is an origin centered circle.*

Recal the Maclaurin series for sin*:*

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Therefore

$$\begin{aligned} z^2 \sin(1/z) &= z^2 \sum n = 0^{\infty} (-1)^n \frac{1}{z^{2n+1}(2n+1)!} \\ &= z^2 \left(\frac{1}{z \cdot 1!} - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} - \frac{1}{z^7 \cdot 7!} + \cdots \right) \\ &= z - \frac{1}{z \cdot 3!} + \frac{1}{z^3 \cdot 5!} - \frac{1}{z^5 \cdot 7!} + \cdots \end{aligned}$$

From this we see that $\operatorname*{Res}_0 = \frac{-1}{3!} = \frac{-1}{6}$*. So,*

$$\int_C z^2 \sin(1/z) dz = 2\pi i \left(\frac{-1}{6} \right) = \frac{-\pi i}{3}.$$

Example 18.7. *Find the integral* $\int_C e^{1/z^2} dz$ *where* C *is an origin centered circle.*

Firstly, recall the familiar Maclaurin series for e^z*:*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

From this we have

$$\begin{aligned} e^{1/z^2} &= \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} \\ &= 1 + \frac{1}{z^2 \cdot 1!} + \frac{1}{z^4 \cdot 2!} + \frac{1}{z^4 \cdot 3!} + \cdots . \end{aligned}$$

Therefore $\operatorname*{Res}_0 = 0$*, and*

$$\int_C e^{(1/z^2)} dz = 0.$$

Example 18.8. *Find the integral* $\int_C \frac{1}{z(z-2)^4} dz$ *where* C *is a circle centered at 2 with radius 1.*

There are singularities $z = 0$ *and* $z = 2$*. However* $z = 2$ *is the only singularity inside our contour.*

First we will find the power series for $\frac{1}{z}$ centered at 2:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2+z-2} \\ &= \frac{1}{2}\cdot\frac{1}{1-\left(\frac{-(z-1)}{2}\right)} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-(z-2)}{2}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^{n+1}}.\end{aligned}$$

And from this

$$\begin{aligned}\frac{1}{z(z-2)^4} &= \frac{1}{(z-2)^4}\left(\frac{1}{2}-\frac{(z-2)^2}{2^2}+\frac{(z-2)^2}{2^3}-\frac{(z-2)^3}{2^4}+\cdots\right) \\ &= \frac{1}{2(z-2)^4}-\frac{1}{2^2(z-2)^3}+\frac{1}{2^3(z-2)^2}-\frac{1}{2^4(z-2)}+\frac{1}{2^5}-\cdots.\end{aligned}$$

So, $\operatorname{Res}_2\left(\frac{1}{z(z-2)^4}\right)=\frac{-1}{2^4}=\frac{-1}{16}$. *And*

$$\int_C \frac{1}{z(z-2)^4}dz = 2\pi i\frac{-1}{16} = \frac{-\pi i}{8}.$$

Theorem 18.9 (Cauchy Residue Theorem)**.** *Let C be a simple, closed, positively oriented contour. If $f(z)$ is analytic on C, and analytic inside C except for a finite number of points z_k, then*

$$\int_C f(z)dz = 2\pi i\sum_k \operatorname{Res}_{z_k}(f).$$

This theorem can be proved using deformation of contours.

Example 18.10. *Find the integral $\int_C \frac{5z-2}{z(z-1)}dz$ where C is an origin centered circle with radius 2.*

We will want to find Res_0 and Res_1. First lets find Res_0. To do this we find the power series of $\frac{1}{z-1}$:

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Then from this we have

$$\begin{aligned}\frac{5z-2}{z(z-1)} &= \left(\frac{5z-2}{z}\right)\left(-\sum_{n=0}^{\infty} z^n\right)\\ &= (5-\frac{2}{z})(-1-z-z^2-z^4-\cdots)\\ &= (-5-5z-5z^2-5z^3-\cdots)+(\frac{2}{z}+2+2z+2z^2+2z^3+\cdots)\end{aligned}$$

and we see that $\operatorname*{Res}_0 = 2$.

We now find $\operatorname*{Res}_1$. *We have the following power series*

$$\frac{1}{z} = \frac{1}{1+z-1} = \sum_{n=0}^{\infty}(-(z-1))^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$

Also

$$\frac{5z-2}{z-1} = \frac{5z-5+3}{z-1} = 5+\frac{3}{z-1}.$$

From these we have

$$\begin{aligned}\frac{5z-2}{z(z-1)} &= \left(5+\frac{3}{z-1}\right)\left(1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right)\\ &= \left(5-5(z-1)+5(z-1)^2-5(z-1)^3+\cdots\right)\\ &\quad+\left(\frac{3}{z-1}-3+3(z-1)-3(z-1)^2+\cdots\right).\end{aligned}$$

Thus $\operatorname*{Res}_1 = 3$.

From the two residues we found we have

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i(2+3) = 10\pi i.$$

18.1. **Partial Fractions.** A tool that often helps in finding residues is partial fraction decomposition. For example in the previous example we could have used partial fractions to find

$$\frac{5z-2}{z(z-1)} = \frac{2}{z}+\frac{3}{z-1}.$$

Immediately from this we see that the residues are 2 and 3.

19. Lecture 19: Singularities

19.1. Types of Singularities. Say we have the Laurent series

$$f(z) = \cdots + \frac{C_{-3}}{(z-z_0)^3} + \frac{C_{-2}}{(z-z_0)^2} + \frac{C_{-1}}{(z-z_0)} + \sum_{n=0}^{\infty} C_n (z-z_0)^n.$$

- If *lowest* power term is C_{-m} we say that there is a **Pole of Order m**.
- A pole of order 1 is also called a **Simple Pole**.
- If infinitely many negative power terms we say that we have an **Essential Singularity**.
- If there are no negative power terms we have a **Removable Singularity**.

Example 19.1. *What are the singularities of* $f(z) = \frac{e^{1/z}}{(z-1)^3(z+5)}$*?*
The singularities are
- *An essential singularity* $z = 0$.
- *A pole of order 3 at* $z = 1$.
- *A simple pole at* $z = -5$.

19.2. Residues at Poles.

Theorem 19.2. $f(z)$ *has a pole of order* m *at* z_0 *if and only if*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

for $\phi(z)$ *an analytic function such that* $\phi(z_0) \neq 0$. *Then*

$$\frac{\phi^{m-1}(z_0)}{(z-z_0)!}$$

for a simple pole, $\operatorname*{Res}_{z_0} = \phi(z_0)$.

Proof. (⇒) Assume we have a pole of order m, then

$$\begin{aligned} f(z) &= \frac{C_{-m}}{(z-z_0)^m} + \frac{C_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots \\ &= \frac{1}{(z-z_0)^m} \underbrace{\left(C_{-m} + C_{-(m-1)}(z-z_0) + \cdots\right)}_{\text{Taylor series for } \phi(z).} \end{aligned}$$

And we see that $\phi(z)$ is analytic (since it has no negative powers) and that $\phi(z_0) = C_{-m} \neq 0$.

($\Leftarrow$) We have

$$\begin{aligned} f(z) &= \frac{\phi(z)}{(z-z_0)^m} \\ &= \frac{1}{(z-z_0)^m}\left(d_0 + d_1(z-z_0)^2 + \cdots + d_{m-1}(z-z_0)^{m-1} + d_m(z-z_0)^m + \cdots\right) \\ &= \frac{d_0}{(z-z_0)^m} + \frac{d_1}{(z-z_0)^{m-1}} + \frac{d_2}{(z-z_0)^{m-2}} + \cdots + \frac{d_{m-1}}{(z-z_0)} + d_m + \cdots . \end{aligned}$$

From this we see that z_0 is a pole of order m and also

$$\operatorname*{Res}_{z_0}(f) = d_{m-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

since d_{m-1} is the $m-1$ term in the Taylor series of $\phi(z)$. □

Example 19.3. *Find the residues of* $f(z) = \frac{z+1}{z^2+9}$.

We can rewrite f *as*

$$f(z) = \frac{z+1}{(z-3i)(z+3i)}.$$

This makes it clear that the singularities of $f(z)$ *are* $\pm 3i$. *Finding the residues we have*

$$\operatorname*{Res}_{3i}\left(\frac{(z+1)/(z+3i)}{(z-3i)}\right) = \frac{\phi(3i)}{0!} = \frac{3i+1}{6i}$$

where $\phi(z) = \frac{z+1}{z+3i}$. *Similarly*

$$\operatorname*{Res}_{-3i}\left(\frac{(z+1)/(z-3i)}{z+3i}\right) = \frac{\phi(-3i)}{0!} = \frac{1-3i}{-6i}$$

where $\phi(z) = \frac{z+1}{z-3i}$.

Example 19.4. *Find the residues for* $\frac{\cos(z)}{z^2(z+2)}$.

We have singularities at 0 *and* -2. *The singularity at* 0 *is a pole of order* 2. *Finding its residue*

$$\operatorname*{Res}_{0}\left(\frac{\cos(z)/(z+2)}{z^2}\right) = \frac{\phi'(0)}{1!}$$

where $\phi(z) = \cos(z)/(z+2)$ *and thus*

$$\phi'(z) = \frac{-\sin(z)(z+2) - \cos(z)}{(z+2)^2} \qquad \Rightarrow \qquad \phi'(0) = \frac{-\sin(0)(2) - \cos(0)}{(2)^2}.$$

The other singularity is just a simple pole, so

$$\operatorname*{Res}_{-2}\left(\frac{\cos(z)/z^2}{z+2}\right) = \phi(-2) = \frac{\cos(-2)}{4}$$

where $\phi(z) = \frac{\cos(z)}{z^2}$.

19.3. **zeros.**

Definition 19.5. $f(z)$ is said to have a **zero** of order m at z_0 if

$$f(z_0) = f'(z) = f''(z) = \cdots = f^{(m-1)}(z_0) = 0,$$

but $f^{(m)}(z_0) \neq 0$.

Example 19.6. *z^3 has a zero of order 3 at $z = 0$ since*

$$\begin{aligned} f(z) &= z^3 \to f(0) = 0 \\ f'(z) &= 2z^2 \to f'(0) = 0 \\ f''(z) &= 4z \to f''(0) = 0 \\ f'''(z) &= 4 \to f'''(0) = 4. \end{aligned}$$

Theorem 19.7. *A analytic function $f(z)$ has a zero of order m at z_0 if and only $f(z) = (z - z_0)^m g(z)$ for some analytic function $g(z)$ such that $g(z_0) \neq 0$.*

Proof. We have the Taylor series

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_m(z - z_0)^m + \cdots$$

where $c_m = \frac{f^{(m)}(z_0)}{m!}$. All terms of the taylor sere is up to the term with power $m - 1$ are equal to 0. $\square$

Remark 19.8. This theorem implies the uniqueness theorem. This is because saying $f = g$ on an open set of a line is the same as saying $f - g = 0$. If the zero is of order m, then $f - g = (z - z_0)^n h(z)$ where $h(z)$ is analytic and $h(z_0) \neq 0$.

Theorem 19.9. *Let $p(z)$ and $q(z)$ be analytic at $z = z_0$ and also have $p(z_0) \neq 0$ and $q(z)$ have a zero of order m at z_0. Then $p(z)/q(z)$ has a pole of order m at z_0.*

Example 19.10. *What is the order of the pole at $z = 0$ in $\frac{1}{z(e^z-1)}$? Let $q = z(e^z - 1)$. We know $q(0) = 0$. Also*

$$\begin{aligned} q'(z) &= 1(e^z - 1) + ze^z \to q'(0) = 0 \\ q''(z) &= e^z + e^z + ze^z \to q''(0) = 2. \end{aligned}$$

Thus $\frac{1}{z(e^z-1)}$ has a pole of order 2 at 0.

Theorem 19.11. *Let $p(z)$ and $q(z)$ be analytic at z_0 with $p(z_0) \neq 0$ and $q'(z_0) \neq 0$ (i.e. q is a zero of order 1). Then* $\operatorname*{Res}_{z_0}\left(\frac{p}{q}\right) = \frac{p(z_0)}{q'(z_0)}$.

Proof. We have $q(z) = (z-z_0)g(z)$, then $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Then

$$\frac{p}{q} = \frac{p(z)}{(z-z_0)g(z)}.$$

So by the previous theorem,

$$\frac{p(z)}{q(z)} = \frac{(p(z)/g(z))}{z-z_0}$$

and

$$\operatorname*{Res}_{z_0}\left(\frac{p}{q}\right) = \phi(z_0) = \frac{p(z_0)}{g(z_0)}.$$

But,

$$\begin{aligned} q(z) &= (z-z_0)g(z) \\ q'(z) &= 1 \cdot g(z) + (z-z_0)g'(z), \end{aligned}$$

so $q'(z_0) = g(z_0) + 0$. Thus

$$\operatorname*{Res}_{z_0}(p/q) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}.$$

□

Example 19.12. *Find the residues of* $\frac{z+1}{z^2+9}$.

Firstly note that the singularities are at $\pm 3i$. Using the previous theorem, letting $p = z+1$ and $q = z^2+9$, we have

$$\begin{aligned} \operatorname*{Res}_{3i} &= \frac{p(3i)}{q'(3i)} = \frac{3i+1}{2(3i)} \\ \operatorname*{Res}_{-3i} &= \frac{p(-3i)}{q'(-3i)} = \frac{-3i+1}{2(-3i)}. \end{aligned}$$

If you try this method and $q'(z_0) = 0$ then this means that z_0 is not a simple pole and you will have to use a different method to find the residue.

Example 19.13. *Find* $\operatorname*{Res}_{0}(\cot(z))$.

Using the p/q' method with $p = \cos(z)$ and $q = \sin(z)$,

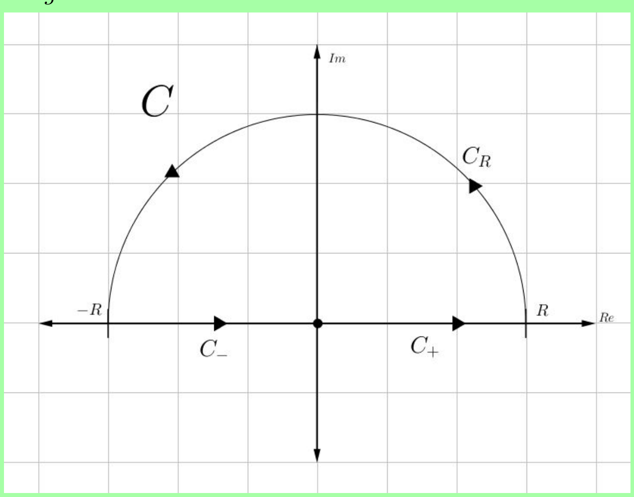
$$\operatorname*{Res}_{0}(\cot(z)) = \operatorname*{Res}_{0}\left(\frac{\cos(z)}{\sin(z)}\right) = \frac{\cos(0)}{\sin'(0)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

20. Lecture 20: Inproper Integrals

In calculus
$$\int_0^\infty f(x)dz = \lim_{R\to\infty}\int_0^R f(x)dx.$$
Similarly
$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_k^R f(x)dx + \lim_{R\to\infty}\int_{-R}^k f(x)dx.$$
We will now give a technique for solving improper integrals which uses complex analysis. This technique is best illustrated thru examples.

Example 20.1. *Find* $\int_0^\infty \frac{x^2}{x^6+1}dx = \lim_{R\to\infty}\int_0^R \frac{x^2}{x^6+1}dx$.

Consider the contour $C = C_+ \cup C_R \cup C_-$ *where* C_+ *is the integral from the origin to* R *along the real axis,* C_R *is the origin centered half circle with radius* R *lying in the upper half-plane, and* C_- *is the contour from* $-R$ *to the origin along the real axis. These contours are illustrated in the following diagram.*



Note that we have the following equality of (complex) integrals,
$$\int_C \frac{z^2}{z^6+1}dz = \int_{C_+}\frac{z^2}{z^6+1}dz + \int_{C_R}\frac{z^2}{z^6+1}dz + \int_{C_-}\frac{z^2}{z^6+1}dz.$$
Also,
$$z^6+1=0 \Rightarrow z^6 = -1 = e^{i(\pi+2\pi k)}$$
$$z = e^{i(\pi/6+2\pi k/6)}.$$
Therefore the singularities of $\frac{z^2}{z^6+1}$ *which are inside* C *are* $e^{i\pi/6}$*,* i*, and* $e^{i5\pi/6}$*. And so*
$$\int_C \frac{z^2}{z^6+1} = 2\pi i\left(\operatorname*{Res}_{e^{i\pi/6}} + \operatorname*{Res}_{i} + \operatorname*{Res}_{e^{i5\pi/6}}\right).$$

Using the p/q' method:

$$\frac{p}{q'} = \frac{z^2}{6z^5} = \frac{1}{6z^3}.$$

Substituting this into our formula for the integral around C,

$$\begin{aligned}\int_C \frac{z^2}{z^6+1} &= 2\pi i\left(\frac{1}{6e^{i3\pi/6}} + \frac{1}{6i^3} + \frac{1}{6e^{i15\pi/6}}\right)\\ &= 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right)\\ &= \frac{\pi}{3}.\end{aligned}$$

Now lets find the integral along C_-. To do this we will use the parametrization $z = -t$ for $0 \le t \le R$ and $dz = (-1)dt$ to obtain

$$\begin{aligned}\int_{C_-} \frac{z^2}{z^6+1}dz = 0\int_{C_-} \frac{(-t)^2}{(-t)^6+1}(-1)dt\\ &= \int_0^R \frac{t^2}{t^6+1}dt\\ &= \int_0^R \frac{x^2}{x^6+1}dx.\end{aligned}$$

Notice that we have the integral on C_- in terms of the integral we want to find. Now we will fin the integral on C_R. We will use the $M \cdot L$ method,

$$\left|\int_{C_R} \frac{z^2}{z^6+1}dz\right| \le M \cdot L$$

for $L = 2\pi$ (the length of C_R. Also, on C_R, $|z^2| = R^2$ and $|z^6+1| \ge \big||z|^6 - 1\big| = R^6 - 1$. Therefore

$$\left|\frac{z^2}{z^6+1}\right| \le \frac{R^2}{R^6-1} = M.$$

Therefore

$$\left|\int_{C_R} \frac{z^2}{z^6+1}dz\right| \le M \cdot L = \frac{\pi R^3}{R^6-1}.$$

And as $R \to \infty$ this expression goes to zero (because the power is higher in the denominator than the numerator). From this we must have

$$\int_{C_R} \frac{z^2}{z^6+1}dz = 0.$$

So we have

$$\begin{aligned}\int_C \frac{z^2}{z^6+1}dz = \int_{C_+} \frac{z^2}{z^6+1}dz + \int_{C_R} \frac{z^2}{z^6+1}dz + \int_{C_-} \frac{z^2}{z^6+1}dz\\ \frac{\pi}{3} = 2\int_0^\infty \frac{x^2}{x^6+1}dx.\end{aligned}$$

So as $R \to \infty$

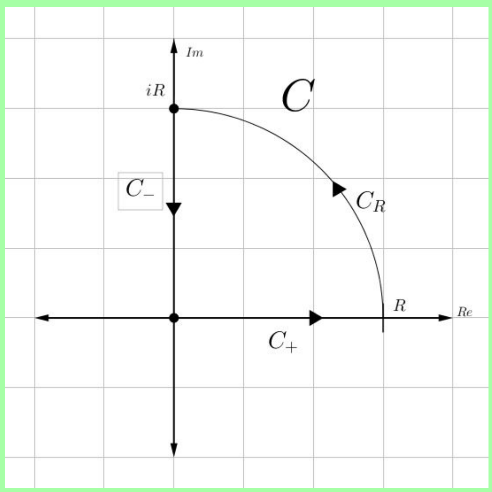
$$\frac{\pi}{3} = 2\int_0^\infty \frac{x^2}{x^6+1}dx.$$

And finally we get an expression for the improper integral

$$\int_0^\infty \frac{x^2}{x^6+1}dx = \frac{\pi}{6}.$$

Example 20.2. *Evaluate* $\int_0^\infty \frac{x}{x^4+1}dx = \lim_{R\to\infty}\int_0^R \frac{x}{x^4+1}dx$.

This will be fairly similar to the previous example except we will use a slightly different contour. We will use $C = C_+ \cup C_R \cup C_-$ *where* C_+ *is the contour from the origin to* R *along the real axis,* C_R *is the quarter circle in the first quadrant of the complex plane, and* C_- *is the contour from* iR *to the origin along the imaginary axis. These contours are illustrated in the following diagram.*



From these contours we have

$$\int_C \frac{z}{z^4+1}dz = \int_{C_+} \frac{z}{z^4+1}dz + \int_{C_R} \frac{z}{z^4+1}dz + \int_{C_-} \frac{z}{z^4+1}dz.$$

Since

$$z^4+1 = 0$$
$$z^4 = -1 = e^{i(\pi+2\pi k)}$$
$$z = e^{i(\pi/4+2\pi k/4)}$$

the singularities of $\frac{z}{z^4+1}$ *are* $z = e^{i(\pi/4+2\pi k/4)}$. *However only 1 of these 4 singularities is inside* C, *namely* $e^{i\pi/4}$. *So* $\int_C \frac{z}{z^4+1}dz = 2\pi i \operatorname*{Res}_{e^{i\pi/4}}$. *Using the* p/q' *method,*

$$\frac{p}{q'} = \frac{z}{4z^3} = \frac{1}{4z^2}$$

So

$$\int_C \frac{z}{z^4+1} = 2\pi i \operatorname*{Res}_{e^{i\pi/4}}$$
$$= 2\pi i \left(\frac{1}{4\left(e^{i\pi/4}\right)^2}\right)$$
$$= 2\pi i \frac{1}{4e^{i\pi/2}}$$
$$= \frac{2\pi i}{4i}$$
$$= \frac{\pi}{2}.$$

Now lets find our integral along C_-. To do this we can parametrize C_- as $z = it$ for $0 \leq t \leq R$. Using this parametrization we have

$$\int_{C_-} \frac{z}{z^4+1}dz = -\int_0^R \frac{it}{(it)^4+1}idt = \int_0^R \frac{t}{t^4+1}dt = \int_0^R \frac{x}{x^4+1}dx.$$

Now lets find the integral along the contour C_R. We can find a bound

$$\left|\int_{C_R} \frac{z}{z^4+1}dz\right| = M \cdot L$$

where $L = \frac{\pi}{2}R$ (the length of C_R). Also $|z| + R$ and $|z^4+1| \geq |z|^4 - |1| = R^4 - 1$. From this we can take M to be $\frac{R}{R^4-1}$. Therefore

$$\left|\int_{C_R} \frac{z}{z^4+1}dz\right| \leq M \cdot L = \frac{R^2\pi}{2R^4-2}$$

which equals zero $R \to \infty$. So from

$$\int_C \frac{z}{z^4+1}dz = \int_{C_+} \frac{z}{z^4+1}dz + \int_{C_R} \frac{z}{z^4+1}dz + \int_{C_-} \frac{z}{z^4+1}dz$$

$$\frac{\pi}{2} = 2\int_0^R \frac{z}{z^4+1}dz + \int_{C_R} \frac{z}{z^4+1}dz,$$

taking the limit as $R \to \infty$, we have

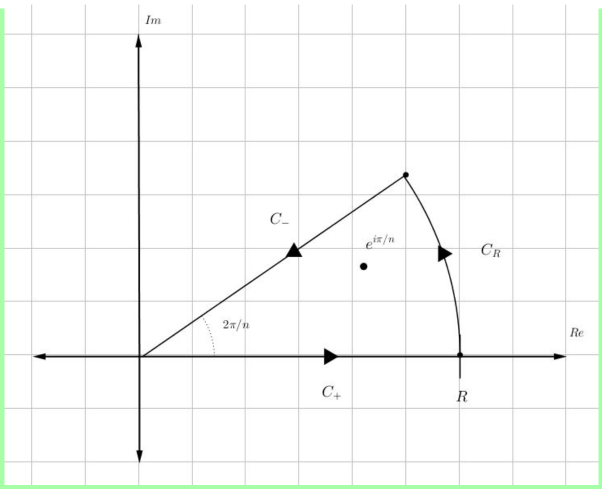
$$\frac{\pi}{2} = 2\int_0^\infty \frac{x}{x^4+1}dx.$$

We find our solution

$$\int_0^\infty \frac{x}{x^4+1}dx = \frac{\pi}{4}.$$

Example 20.3. *Evaluate $\int_0^\infty \frac{1}{x^n+1}dx$ for an integer $n \geq 2$.*

Use the contour $C = C_+ \cup C_R \cup C_-$ as illustrated here:



Then we will have the equation

$$\int_C \frac{1}{z^n+1}dz = \int_{C_+} \frac{1}{z^n+1}dz + \int_{C_R} \frac{1}{z^n+1}dz + \int_{C_-} \frac{1}{z^n+1}dz.$$

The singularities of $\frac{1}{z^n+1}$ *are the zs satisfying* $z^n = -1$*, which are* $z = e^{\pi/n + 2\pi k/n}$*. Only one of these n singularities will end up being inside C; namely* $e^{i\pi/n}$*. By the* p/q' *method we have* $\frac{p}{q'} = \frac{1}{nz^{n-1}}$*. And so*

$$\int_C \frac{1}{z^n+1}dz = 2\pi i \left(\frac{1}{ne^{\frac{i\pi(n-1)}{n}}} \right) = \frac{2\pi i}{n} \cdot \frac{1}{e^{i\pi - \frac{1\pi}{n}}} = \frac{-2\pi i}{n} e^{\frac{i\pi}{n}}.$$

Now lets find $\int_{C_-} \frac{1}{z^n+1}dz$*. We can parametrize* C_- *as* $z = te^{i2\pi/n}$ *for* $0 \leq t \leq R$ *and thus* $dz = e^{i2\pi/n}dt$*. Using this parametrization we can compute the integral*

$$\begin{aligned} \int_{C_-} \frac{1}{z^n+1}dz &= -\int_0^R \frac{1}{\left(te^{i2\pi/n}\right)^n + 1} e^{i2\pi/n} dt \\ &= -\int_0^R \frac{1}{t^n+1} e^{\frac{i2\pi}{n}} dt \\ &= -e^{\frac{i2\pi}{n}} \int_0^R \frac{1}{x^n+1} dx. \end{aligned}$$

So we have

$$\begin{aligned} &\int_C \frac{1}{z^n+1}dz = 2\int_{C_+} \frac{1}{z^n+1}dz + \int_{C_R} \frac{1}{z^n+1}dz + \int_{C_-} \frac{1}{z^n+1}dz \\ \Rightarrow \quad &\frac{-2\pi i}{n} e^{i\pi/n} = \left(1 - e^{i2\pi/n}\right) \int_0^R \frac{1}{x^n+1}dx + \int_{C_R} \frac{1}{z^n+1}dz. \end{aligned}$$

So we must find $\int_{C_R} \frac{1}{z^n+1}dz$*. Firstly notice*

$$\left| \int_{C_R} \frac{1}{z^n+1}dz \right| \leq M \cdot L$$

where $L = 2\pi R/n$ *and*

$$\left|\frac{1}{z^n+1}\right| \leq \frac{1}{|z|^n - 1} = \frac{1}{R^n - 1} = M.$$

Then

$$\left|\int_{C_R} \frac{1}{z^n+1} dz\right| \leq M \cdot L = \frac{2\pi R}{n(R^n-1)}$$

However, if $n > 1$ *then as* $R \to \infty$ *this goes to zero. (Recall* $n \geq 2$ *in our example so this goes to zero). So then we must have (as* $R \to \infty$*)*

$$\frac{-2\pi i}{n} e^{i\pi/n} = \left(1 - e^{i2\pi/n}\right) \int_0^R \frac{1}{x^n+1} dx + \int_{C_R} \frac{1}{z^n+1} dz$$

$$\Rightarrow \quad \frac{-2\pi i}{n} e^{i\pi/n} = \left(1 - e^{i2\pi/n}\right) \int_0^\infty \frac{1}{x^n+1} dx + 0.$$

Therefore

$$\begin{aligned}
\int_0^\infty \frac{1}{x^n+1} dx &= \frac{-2\pi i e^{i\pi/4}}{n(1 - e^{i2\pi/n}} \\
&= \frac{-2\pi i e^{i\pi/n}}{n(1 - e^{i2\pi/n})} \\
&= \frac{2\pi i e^{i\pi/n}}{n\left(e^{i\pi/n} - 1\right)} \\
&= \frac{2\pi i}{n\left(e^{i\pi/n} - e^{-i\pi/n}\right)} \\
&= \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)}.
\end{aligned}$$

21. Lecture 21: Jordan's Lemma and Inproper Integrals

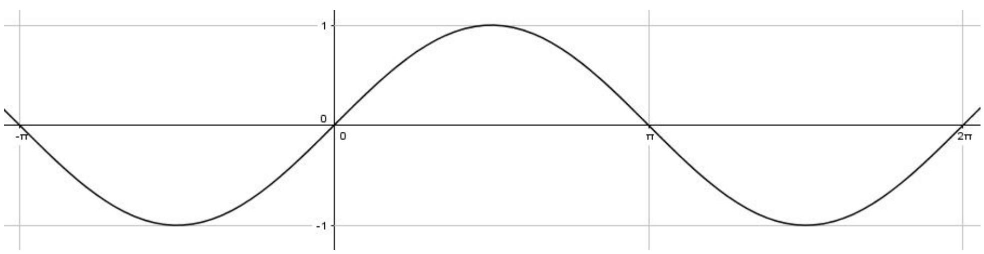
21.1. Jordan's Lemma.

Theorem 21.1 (Jordan's Inequality)**.**

$$\int_0^\pi e^{-k\sin\theta} d\theta < \frac{\pi}{k}$$

for $k > 0$*.*

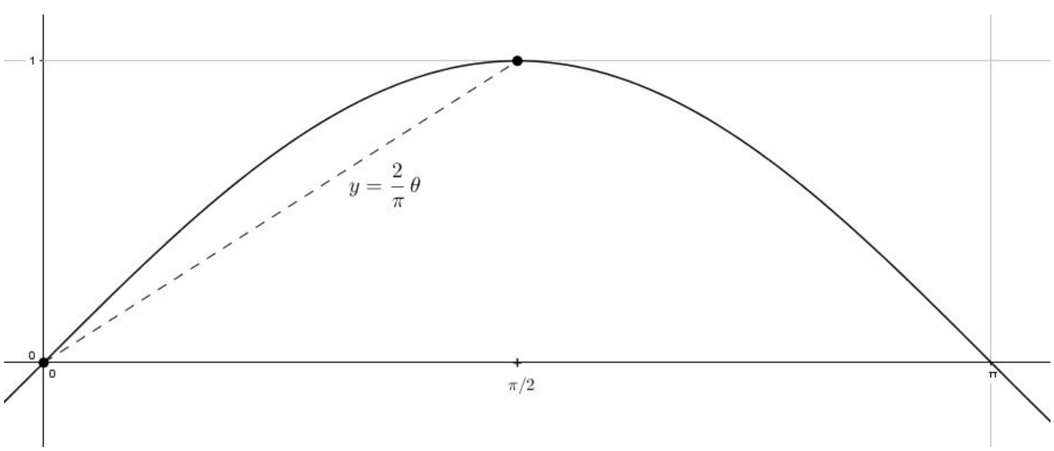
Proof. We want to replace $-k\sin\theta$ with something we can integrate. Recall what the curve $\sin\theta$ looks like:



Also notice

$$\int_0^\pi e^{-k\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-k\sin\theta} d\theta.$$

Consider the line $y = \frac{2}{\pi}\theta$. This line, illustrated as the dashed line in the following diagram, is a lower bound for $\sin\theta$.



That is $\frac{2}{\pi}\theta \leq \sin\theta$ for $0 \leq \theta \leq \pi/2$. Thus $-k\frac{2}{\pi}\theta \geq -k\sin\theta$ and $e^{-k\frac{2}{\pi}\theta} \geq e^{-k\sin\theta}$. So,

$$\begin{aligned} 2\int_0^{\pi/2} e^{-k\sin\theta}d\theta &< 2\int_0^{\pi/2} e^{-k\frac{2}{k}\frac{2}{\pi}\theta}d\theta \\ &< 2\left(\frac{-\pi}{2k}e^{\frac{-k2}{\pi}\theta}\Big|_0^{\frac{\pi}{2}}\right) \\ &< \frac{-\pi}{k}e^{-k} + \frac{\pi}{k}\cdot 1 \\ &< \frac{\pi}{k}. \end{aligned}$$

□

Theorem 21.2 (Jordan's Lemma)**.** *Let $f(z)$ be an analytic function in the upper half plane outside of $|z| = R$. If...*

- *$C_R = z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$.*
- *On C_R, $|f(z)| \leq M_R$ and $\lim_{R\to\infty} M_R = 0$.*
- *a is a positive real number*

Then,

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iaz}f(z)dz\right| = 0.$$

Proof. We have

$$\left|\int_{C_R} e^{iaz}f(z)dz\right| = \left|\int_0^{\pi} e^{iaRe^{i\theta}}f(Re^{i\theta})iRe^{i\theta}d\theta\right|.$$

But,

$$iaRe^{i\theta} = iaR(\cos\theta + i\sin\theta) = iaR\cos\theta - aR\sin\theta.$$

Thus

$$\left|\int_0^{\pi} e^{iaRe^{i\theta}}f(Re^{i\theta})iRe^{i\theta}d\theta\right| = \left|\int_0^{\pi} e^{iaR\cos\theta}e^{-iaR\sin\theta}f(Re^{i\theta})iRe^{i\theta}d\theta\right|.$$

Then we have the inequality

$$\left|\int_0^{\pi} e^{iaR\cos\theta}e^{-iaR\sin\theta}f(Re^{i\theta})iRe^{i\theta}d\theta\right| \leq \int_0^{\pi} 1\cdot e^{-a\sin\theta}M_R R d\theta = M_R R\int_0^{\pi} e^{-aR\sin\theta}d\theta.$$

By Jordan's inequality

$$M_R R\int_0^{\pi} e^{-aR\sin\theta}d\theta \leq M_R R\frac{\pi}{aR} = M_R\frac{\pi}{a}.$$

But now

$$\lim_{R\to\infty}\left|\int e^{iaz}f(z)\right| \leq \lim_{R\to\infty} M_R\frac{\pi}{a} = 0.$$

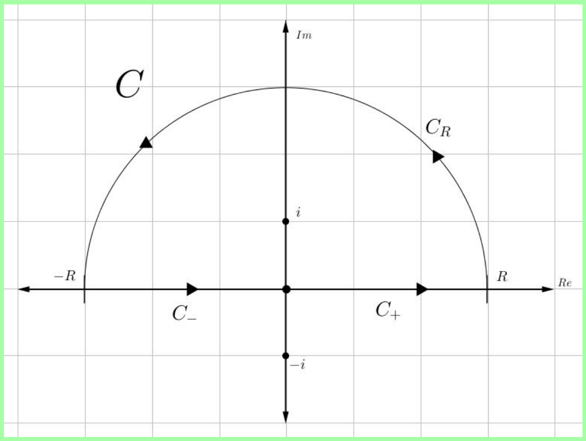
□

Example 21.3. *Calculate* $\int_0^\infty \frac{\cos(x)}{1+x^2}dx = \lim_{R\to\infty}\int_0^R \frac{\cos(x)}{1+x^2}dx$.

We consider the complex integral

$$\int_C \frac{e^{iz}}{1+z^2}dz = \int_{C_+}\frac{e^{iz}}{1+z^2}dz + \int_{C_R}\frac{e^{iz}}{1+z^2}dz + \int_{C_-}\frac{e^{iz}}{1+z^2}dz$$

where $C = C_+ \cup C_R \cup C_-$ *as illustrated:*



We have singularities at $\pm i$*, but only* i *is inside our contour* C*. Finding the residue we have* $\operatorname*{Res}_i = p(i)/q'(i)$ *where*

$$\frac{p}{q} = \frac{e^{iz}}{1+z^2} \quad \Rightarrow \quad \frac{p}{q'} = \frac{e^{iz}}{ez} \quad \Rightarrow \quad \operatorname*{Res}_i = \frac{e^{i\cdot i}}{2i} = \frac{1}{2ie}.$$

So,

$$\int_C \frac{e^{iz}}{1+z^2}dz = 2\pi i \operatorname*{Res}_i = \frac{\pi}{e}.$$

We also have

$$\int_{C_+}\frac{e^{iz}}{1+z^2}dz = \int_0^R \frac{e^{ix}}{1+x^2}dx.$$

Additionally by parametrizeing C_- *as* $z = -t$ *for* $0 \le t \le R$ *we have*

$$\int_{C_-}\frac{e^{iz}}{1+z^2}dz = -\int_0^R \frac{e^{i(-t)}}{1+(-t)^2}(-1)dt = \int_0^R \frac{e^{-it}}{1+t^2}dt.$$

For the integral along C_R *note*

$$\lim_{R\to\infty}\left|\frac{1}{1+z^2}\right| \le \lim_{R\to\infty}\frac{1}{R^2-1} = 0.$$

Thus we can apply Jordan's Lemma to obtain

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iz}\frac{1}{1+z^2}\right| = 0.$$

From these integrals we have just found we have the equation

$$\frac{\pi}{e} = \int_0^R \frac{e^{ix}}{1+x^2}dx + \int_0^R \frac{e^{-ix}}{1+x^2}dx + \int_{C_R}\frac{e^{iz}}{1+z^2}dz.$$

Taking the limit of this equation as $R \to \infty$,

$$\frac{\pi}{e} = \int_0^\infty \frac{e^{ix} + e^{-ix}}{1+x^2} dx + 0.$$

But notice $e^{ix} + e^{-ix} = (\cos(x) + i\sin(x)) + (\cos(x) - i\sin(x)) = 2\cos(x)$, *thus*

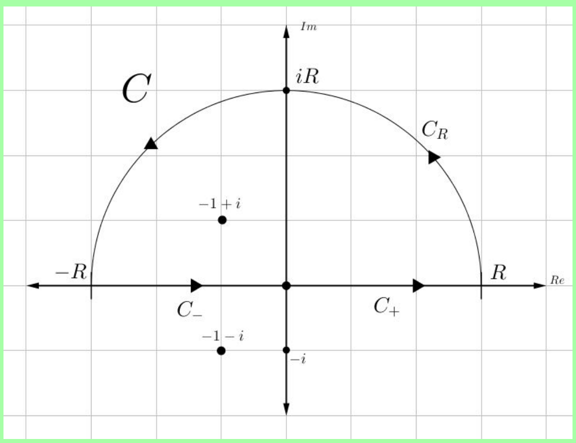
$$\frac{\pi}{e} = \int_0^\infty \frac{2\cos(x)}{1+x^2} dx.$$

And from this we get our solution

$$\int_0^\infty \frac{\cos(x)}{1+x^2} dx = \frac{\pi}{2e}.$$

Example 21.4. $\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(x)}{x^2+2x+2} dx$.

Consider the following contour:



Then we have the equation of integrals

$$\int_C \frac{ze^{iz}}{z^2+2x+2} dz = \int_{-R}^{R} \frac{xe^{ix}}{x^2+2x+2} dx + \int_{C_R} \frac{ze^{iz}}{z^2+2z+2} dz.$$

The solutions to $4z^2+2z+2 = 0$ *are* $z = -1 \pm i$, *which are the singularities of* $\frac{ze^{iz}}{z^2+2x+2}$. *However, only* $-1+i$ *is in* C. *We cand find* $\operatorname*{Res}_{-1+i}$ *with the* p/q' *method:*

$$\frac{p}{q'} = \frac{ze^{iz}}{2z+2},$$
$$\operatorname*{Res}_{-1+i} = \frac{(-1+i)e^{i(-1+i)}}{2(-1+i)+2} = \frac{(-1+i)e^{-i-1}}{2i}.$$

It follows that

$$\int_C \frac{ze^{iz}}{z^2+2z+2} = 2\pi i \left(\frac{(-1+i)e^{-i-1}}{2i} \right) < \pi(-1+i)e^{-i-1}.$$

To find the integral on C_R first consider the inequality

$$|z^2+2z+2| \geq R^2-2R-2.$$

So

$$\left|\frac{z}{z^2+2z+2}\right| \leq \frac{R}{R^2-2R-2}.$$

But $\lim_{R\to\infty}\frac{R}{R^2-2R-2}=0$, *since the power of R in the denominator is higher then the power of R in the numerator. Thus by Jordan's Lemma we can find that*

$$\lim_{R\to\infty}\int_{C_R} e^{iz}\frac{z}{z^2+2z+2}=0.$$

From the integrals we've found we have

$$\pi(-1+i)e^{-i-1}=\int_{-R}^{R}\frac{xe^{ix}}{x^2+2x+2}+\int_{C_R}\frac{ze^iz}{z^2+2z+2}dz.$$

Takeing the limit of this equation as $R\to\infty$,

$$\pi(-1+i)e^{-i-1}=\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{x\cos(x)+i\sin(x)}{x^2+2x+2}dx.$$

With some algebra we have

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x\cos(x)}{x^2+2x+2}dx+i\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^2+2x+2}dx &= \frac{\pi}{e}(-\cos(1)+\sin(1))\\ &= i\frac{\pi}{e}(\cos(1)+\sin(1)).\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^2+2x+2}dx=\frac{\pi}{e}(\cos(1)+\sin(1)).$$

Example 21.5. *Compute the integral* $\int_0^\infty\frac{\cos(x)}{(x^2+4)^2}dx$. *We will use the contour $C=C_+\cup C_R\cup C_-$ where C_+ is the contour from the origin to R along the real line, C_R is the origin centered half-circle contour with radius R, and C_- is the contour from $-R$ to the origin along the real axis. Considering this contour we have the equation*

$$\int_C\frac{e^{iz}}{(z^2+4)^2}dz=\int_{C_+}\frac{e^{iz}}{(z^2+4)^2}dz+\int_{C_R}\frac{e^{iz}}{(z^2+4)^2}dz=\int_{C_-}\frac{e^{iz}}{(z^2+4)^2}dz.$$

Notice that the zeros of z^2+4 coincide with the singularities of $\frac{e^{iz}}{(z^2+4)^2}$ *and are $\pm 2i$; however only $2i$ is in our contour. Now we will use the ϕ method of finding the residue at $2i$. Doing some algebra,*

$$\frac{e^{iz}}{(z^2+4)^2}=\frac{e^{iz}}{(z-2i)^2(z+2i)^2}=\frac{e^{iz}/(z+2i)^2}{(z-2i)^2}$$

Let $\phi(z)=\frac{e^{iz}}{(z+2i)^2}$. *Then*

$$\phi'(z)=\frac{ie^{iz}(z+2i)-2e^{iz}}{(z+2i)^3}.$$

Thus

$$\operatorname*{Res}_{2i} = \phi'(2i) = \frac{ie^{-2}(4i) - 2e^{-2}}{(4i)^3} = \frac{-6e^{-2}}{-64i}.$$

And thus

$$\int_C \frac{e^{iz}}{(z^2+4)^2}dz = 2\pi i \operatorname*{Res}_{2i} = 2\pi i\left(\frac{-6e^{-2}}{-64i}\right) = \frac{\pi 6e^{-2}}{32} = \frac{3\pi}{16e^2}.$$

For the integral about the contour C_+,

$$\int_{C_+} \frac{e^{iz}}{(z^2+4)^2}dz = \frac{e^{iz}}{(x^2+4)^2}dx.$$

For the integral around C_- *we use the parametrization* $z = -t$ *for* $0 \leq t \leq R$. *Then*

$$\int_{C_-} \frac{e^{iz}}{(z^2+4)^2}dz = \int_0^R \frac{e^{-t}}{(t^2+4)^2}dt.$$

For the integral on C_R *notice that*

$$\left|\frac{1}{(z^2+4)^2}\right| \leq \frac{1}{(R-4)^2}.$$

But, $\lim_{R\to\infty} \frac{1}{(R-4)^2} = 0$, *since clearly the denominator is growing faster than the (constant) numerator. Thus by and application of Jordans lemma we can find*

$$\lim_{R\to\infty} \int_{C_R} e^{iz}\frac{1}{(z^2+4)^2}dz = 0.$$

Therefore, combining the integrals we have found, we have

$$\frac{3\pi}{16e^2} = \int_0^R \frac{e^{ix} + e^{-x}}{(x^2+4)^2}dx + \int_{C_R} \frac{e^{iz}}{(z^2+4)^2}dz.$$

Taking the limit of this equation as $\mathbb{R} \to \infty$,

$$\frac{3\pi}{16e^2} = \int_0^\infty \frac{2\cos(x)}{(x^2+4)^2}dz.$$

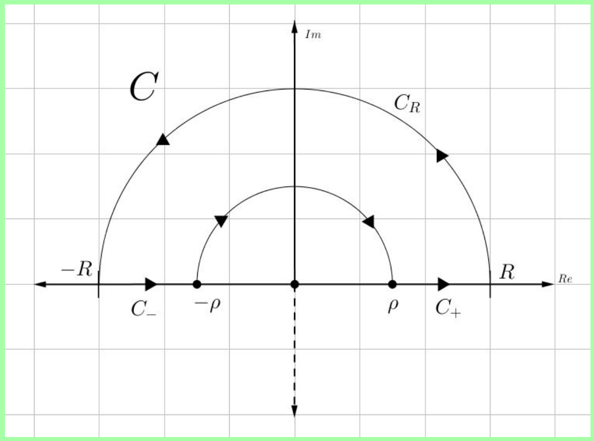
In particular, this implies

$$\int_0^\infty \frac{\cos(x)}{(x^2+4)^2}dx = \frac{3\pi}{32e^2}.$$

22. Lecture 22: More Inproper Integrals

Example 22.1. *Compute the integral* $\int_0^\infty \frac{\ln(x)}{x^2+4}dx$.

We will consider the complex integral $\int_\rho^R \frac{\log(z)}{z^2+4}dz$, *using the branch* $-\pi/2 < \theta < 3\pi/2$. *We will use the contour* $C = C_+ \cup C_R \cup C_- \cup C_\rho$:



From this contour we have the equation

$$\int_C \frac{\log(z)}{(z^2+4)^2}dz = \int_{C_+} \frac{\log(z)}{(z^2+4)^2}dz + \int_{C_R} \frac{\log(z)}{(z^2+4)^2}dz + \int_{C_-} \frac{\log(z)}{(z^2+4)^2}dz + \int_{C_\rho} \frac{\log(z)}{(z^2+4)^2}dz.$$

Rewriting $\frac{\log(z)}{(z^2+4)^2}$ *as* $\frac{\log(z)}{(z-2i)^2(z+2i)^2}$, *we see that the singularities of this function lie at* $\pm 2i$ *with only* $2i$ *inside the contour* C. *Using the* ϕ *method to find the residue*

$$\operatorname*{Res}_{2i} = \operatorname*{Res}_{2i}\left(\frac{\log(z)}{(z+2i)^2}\right) = \phi'(2i)$$

where $\phi(z) = \log(z)/(z+2i)^2$ *and thus*

$$\phi'(z) = \frac{(1/z)(z+2i) - \log(z)\cdot 2\cdot(z+2i)}{(z+2i)^4}.$$

Therefore

$$\begin{aligned}\operatorname*{Res}_{2i} = \phi'(2i) &= \frac{(1/2i)(2i+2i) - \log(2i)\cdot 2\cdot(2i+2i)}{(2i+2i)^4}\\ &= \frac{2-2(\ln(2)+i\pi/2)}{-64i}\\ &= \frac{1-(\ln(2)+i\pi/2)}{-32i}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \frac{\log(z)}{z^2+4)^2}dz &= 2\pi i \operatorname*{Res}_{2i} \\ &= 2\pi i\left(\frac{1-\ln(2)-i\pi/2}{-32i}\right) \\ &= \frac{\pi(2-2\ln(2)-2+i\pi)}{-32} \\ &= \frac{\pi(2\ln(2)-2+i\pi)}{32}.\end{aligned}$$

For the integral on C_+ *we have*

$$\int_{C_+} \frac{\log(z)}{(z^2+4)^2} = \int_\rho^R \frac{\ln(z)}{x^2+4)^2}dx.$$

For the integral on C_- *we parametrize* C_- *as* $z=-t$ *for* $\rho \le t \le R$. *Then*

$$\int_{C_-} \frac{\log(z)}{(z^2+4)^2}dz = -\int_\rho^R \frac{\log(-t)}{((-t)^2+4)^2}(-1)dt = \int_\rho^R \frac{\log(-t)}{(t^2+4)^2}dt.$$

And noticing that $\log(-t) = \ln(t) + i\pi$,

$$\int_\rho^R \frac{\log(-t)}{(t^2+4)^2}dt = \int_\rho^R \frac{\log(t)+i\pi}{(t^2+4)^2}dt.$$

To find our integral on C_R *we consider the bound*

$$\left|\int_{C_R} \frac{\log(z)}{(z^2+4)^2}dz\right| \le M \cdot L$$

where $L = \pi R$ *is the length of the contour and from*

$$|(z+4^2| \ge (R^2-4)^2 \qquad \textit{and} \qquad |log(z)| = |\ln((R)+i\theta| \le \ln(R)+\pi$$

we have

$$M = \frac{\ln(R)+\pi}{(R^2-4)^2}.$$

Thus

$$\left|\int_{C_R} \frac{\log(z)}{(z^2+4)^2}dz\right| \le \frac{\pi R\ln(R)+\pi^2 R}{(R^2-4)^2}.$$

And taking the limit of this inequality as $R \to \infty$:

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{\log(z)}{(z^2+4)^2}dz\right| &\le \lim_{R\to\infty} \frac{\pi R\ln(R)+\pi^2 R}{R^4-8R^2+16} \\ &\le \lim_{R\to\infty} \frac{\pi\ln(R)+\pi^2}{4R^3-16R} \\ &\le \lim_{R\to\infty} \frac{\pi/R}{12R^2-16} = 0\end{aligned}$$

In particular this implies

$$\lim_{R\to\infty}\int_{C_R} \frac{\log(z)}{(z^2+4)^2}dz = 0.$$

For the integral on C_ρ consider the inequality

$$\left| \int_{C_\rho} \frac{\log(z)}{(z^2+4)^2} dz \right| \leq M \cdot L$$

for $L = \pi\rho$ and

$$M = \frac{\pi - \ln \rho}{(4-\rho^2)^2} \leq \frac{\log(z)}{(z^2+4)^2}.$$

Using these L and M we get

$$\left| \int_{C_\rho} \frac{\log(z)}{(z^2+4)^2} dz \right| \leq \frac{\pi^2\rho - \pi\rho\ln(\rho)}{(4-\rho^2)^2}.$$

But, from repeated use of L'Hospital's rule,

$$\begin{aligned} \lim_{\rho\to 0^+} \rho\ln(\rho) &= \lim_{\rho\to 0^+} \frac{\ln(\rho)}{\rho^{-1}} \\ &= \lim_{\rho\to 0^+} \frac{\rho^{-1}}{-\rho^{-2}} \\ &= \lim_{\rho\to 0^+} -\rho = 0. \end{aligned}$$

Using this we have

$$\lim_{\rho\to 0^+} \frac{\pi^2\rho - \pi\rho\ln(\rho)}{(4-\rho^2)^2} = \lim_{\rho\to 0^+} \frac{0-0}{(4-0)^2} = 0.$$

Combining the integrals we've found we have

$$\pi\left(\frac{2\ln(2) - 2 + i\pi}{32}\right) = \int_\rho^R \frac{2\ln(x)+i\pi}{(x^2+4)^2}dx + \int_{C_R} \frac{2\ln(x)+i\pi}{(x^2+4)^2}dx + \int_{C_\rho} \frac{2\ln(x)+i\pi}{(x^2+4)^2}dx.$$

Taking the limit of this expression as $R\to\infty$ and $\rho\to 0$ we get

$$\frac{2\pi\ln(2)-2}{32} + i\frac{\pi^2}{32} = \int_0^\infty \frac{2\ln(x)+i\pi}{(x^2+4)^2}dx.$$

So,

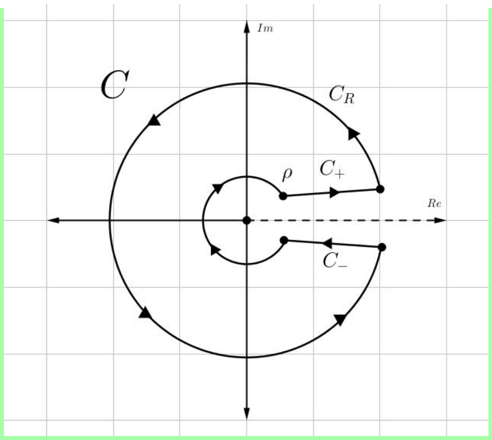
$$\frac{2\pi\ln(2)-2}{32} = \int_0^\infty \frac{2\ln(x)}{(x^2+4)^2}dx.$$

This implies

$$\frac{\pi\ln(2)-\pi}{32} = \int_0^\infty \frac{\ln(x)}{(x^2+4)^2}dx.$$

Example 22.2. *Compute $\int_0^\infty \frac{z^a}{(z+1)^2}dx$ for $a\in\mathbb{R}$.*

We will use the contour $C = C_+ \cup C_- \cup C_R \cup C_\rho$ as illustrated here:



There is a singularity at -1, which is inside of C. Finding its residue,

$$\begin{aligned}\operatorname*{Res}_{-1}\left(\frac{z^a}{(z+1)^a}\right) &= \phi'(-1)\\ &= a(-1)^{a-1}\\ &= ae^{(a-1)(\log(-1)}\\ &= ae^{i\pi(a-1)}\\ &= ae^{0a\pi}e^{-i\pi}\\ &= -ae^{ia\pi}.\end{aligned}$$

So,

$$\int_C \frac{z^a}{(z+1)^2}dz = 2\pi i \operatorname*{Res}_{-1}\left(\frac{z^a}{(z+1)^a}\right) = 2\pi i(-ae^{ia\pi}).$$

For integral on the contour C_+ we have

$$\int_{C_+} \frac{z^a}{(z+1)^2} = \int_\rho^R \frac{x^a}{(x+1)^a}dx.$$

As for the contour C_-, consider the parametrization $z = te^{i2\pi}$. Then

$$\begin{aligned}\int_{C_-} \frac{z^a}{(z+1)^2}dz &= -\int_\rho^R \frac{(e^{i2\pi}t)^a}{(e^{i2\pi}t+1)^2}e^{i2\pi}dt\\ &= -\int_\rho^R \frac{e^{i2\pi a}t^a}{(t+1)^2}dt\\ &= -e^{i2\pi a}\int_\rho^R \frac{x^a}{(x+1)^2}dx.\end{aligned}$$

For the contour C_R consider the inequality

$$\left|\int_{C_R} \frac{z^a}{(z+1)^2}dz\right| \le M \cdot L$$

where $L = 2\pi R$ *and* $M = \frac{R^2}{(R-1)^2}$ *so that*

$$\left|\int_{C_R} \frac{z^a}{(z+1)^2} dz\right| \leq \frac{2\pi R^{a+1}}{(R-1)^2}.$$

But notice that

$$\lim_{R\to\infty} = 0$$

when $a+1 < 2$ *(or equivalently* $a < 1$*). Thus for* $a < 1$

$$\left|\int_{C_R} \frac{z^a}{(z+1)^2} dz\right| \leq 0 \Rightarrow \int_{C_R} \frac{z^a}{(z+1)^2} dz = 0.$$

For C_ρ *we use a similar method to the* C_R *case. We start with the inequality*

$$\left|\int_{C_\rho} \frac{z^a}{(z+1)^2} dz\right| \leq M \cdot L$$

where this time we have $L = 2\pi\rho$ *and* $M = \frac{\rho^a}{(1-\rho)^2}$ *so that* $ML = \frac{2\pi\rho^{a+1}}{(1-\rho)^2}$*. Now we have*

$$\lim_{\rho\to 0} ML = 0$$

when $a+1 > 0$ *(or equivalently* $a > -1$*). Thus for* $a > -1$

$$\left|\int_{C_\rho} \frac{z^a}{(z+1)^2} dz\right| \leq 0 \Rightarrow \int_{C_\rho} \frac{z^a}{(z+1)^2} dz = 0.$$

From the integrals we have found we can deduce that when $-1 < a < a$*, and* $a \neq 0$ *we will have*

$$\int_0^\infty \frac{x^a}{(x+a)^2} dx = \frac{2\pi i(-ae^{ia\pi})}{(1-e^{i2\pi a})} = \frac{\pi a}{\sin(\pi a)}.$$