Solutions - Section 43

(4) C_R is the upper half circle $|z| = R$, counterclockwise. Assume $R > 2$.

$$\begin{aligned}
|2z^2 - 1| &\leq 2|z|^2 + |-1| \\
&\leq 2R^2 + 1 \\
|z^4 + 5z^2 + 4| &= |z^2 + 1| \cdot |z^2 + 4| \\
&\geq ||z|^2 - |1|| \cdot ||z|^2 - |4|| \\
&\geq (R^2 - 1)(R^2 - 4) \\
\left|\frac{2z^2 - 1}{z^2 + 5z^2 + 4}\right| &\leq \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)} \\
L &= \frac{2\pi R}{2} = \pi R \\
\left|\int_{C_R} \frac{2z^2 - 1}{z^2 + 5z^2 + 4} dz\right| &\leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}
\end{aligned}$$

Since the highest power of R on the right side is R^3 in the numerator and R^4 in the denominator, the right side goes to 0 as R goes to infinity. Therefore the integral also goes to 0 as R goes to infinity.

(5) C_R is the circle $|z| = R$, counterclockwise. Assume $R > 1$. Note that $\operatorname{Log} z$ is the principal branch, so $-\pi < \theta < \pi$.

$$\begin{aligned}
|\operatorname{Log} z| &= |\ln R + i\theta| \\
&\leq \ln R + \pi \\
|z^2| &= R^2 \\
\left|\frac{\operatorname{Log} z}{z^2}\right| &\leq \frac{\ln R + \pi}{R^2} \\
L &= 2\pi R \\
\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2} dz\right| &\leq \frac{2\pi R \ln R + 2\pi^2 R}{R^2} \\
\lim_{R \to \infty} \frac{2\pi R \ln R + 2\pi^2 R}{R^2} &= \lim_{R \to \infty} \frac{2\pi \ln R + 2\pi + 2\pi^2}{2R} \\
&= \lim_{R \to \infty} \frac{\frac{2\pi}{R}}{2} = 0
\end{aligned}$$

In the limit evaluation we used l'Hopital's rule twice. Since the modulus of the integral goes to 0 as R goes to infinity, the integral goes to 0 as well.