Solutions - Sections 50, 51, 52

(1a) $f(z) = e^{-z}$ is analytic in C, and $\frac{\pi i}{2}$ is inside C. By the Cauchy integral formula,

$$\int_C \frac{e^{-z}}{z - (\pi i/2)} dz = 2\pi i f(\frac{\pi i}{2}) = 2\pi i e^{-\frac{\pi i}{2}} = (2\pi i)(-i) = 2\pi$$

(1b) $f(z) = \frac{\cos z}{z^2+8}$ is analytic in C since the roots of $z^2 + 8$ are outside C, and 0 is inside C. By the Cauchy integral formula,

$$\int_C \frac{\cos z}{z(z^2+8)} dz = 2\pi i f(0) = (2\pi i)\frac{\cos 0}{0^2+8} = (2\pi i)\frac{1}{8} = \frac{\pi i}{4}$$

(1c) $f(z) = \frac{z}{2}$ is analytic in C, and $-\frac{1}{2}$ is inside C. By the Cauchy integral formula,

$$\int_C \frac{z}{2(z+\frac{1}{2})} dz = 2\pi i f(-\frac{1}{2}) = (2\pi i)\frac{-1}{4} = -\frac{\pi i}{2}$$

(2a) Note that $\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$. $f(z) = \frac{1}{z+2i}$ is analytic in C, the circle $|z-i| = 2$ positively oriented. $2i$ is inside C. By the Cauchy integral formula,

$$\int_C \frac{1}{(z+2i)(z-2i)} dz = 2\pi i f(2i) = 2\pi i \frac{1}{2i+2i} = \frac{\pi}{2}$$

(2b) Note that $\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$. $f(z) = \frac{1}{(z+2i)^2}$ is analytic in C, the circle $|z-i| = 2$ positively oriented. $2i$ is inside C. By the Cauchy integral formula,

$$\int_C \frac{1}{(z+2i)^2(z-2i)^2} dz = \frac{2\pi i}{1!} f'(2i) = 2\pi i \frac{-2}{(2i+2i)^3} = 2\pi i \frac{-2}{-64i} = \frac{\pi}{16}$$

(7) C is the unit circle $z = e^{i\theta}$, $(-\pi \le \theta \le \pi)$. $f(z) = e^{az}$ is analytic in C for all complex constants a and 0 is inside C. By the Cauchy integral formula,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i$$

Writing this integral in terms of θ, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} (ie^{i\theta}) d\theta &= 2\pi i \\
\int_{-\pi}^{\pi} ie^{a\cos\theta + ia\sin\theta} d\theta &= 2\pi i \\
\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta) + i\sin(a\sin\theta)) d\theta &= 2\pi \\
\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) d\theta &= 2\pi \qquad \text{(real part)} \\
\int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta) d\theta &= \pi
\end{aligned}$$

In the last line, we used the fact that the integrand is even.