Solutions - Sections 58, 59

(2) We seek the Taylor series for $f(z) = e^z$ centered at $z = 1$.

Note that since the derivative of e^z is e^z, all derivatives are e^z. That is, $f^{(n)}(z) = e^z$ and $f^{(n)}(1) = e$.

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n \\ &= \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n \\ &= e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \end{aligned}$$

Alternatively, we know the Maclaurin series for $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Therefore

$$\begin{aligned} e^z &= (e)(e^{z-1}) \\ &= e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \end{aligned}$$

(7) We seek the Taylor series for $\frac{1}{1-z}$ centered at $z = i$. We know the Maclaurin series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$.

$$\begin{aligned} \frac{1}{1-z} &= \frac{1}{(1-i)-(z-i)} \\ &= \frac{1}{1-i}\frac{1}{1-\frac{z-i}{1-i}} \\ &= \frac{1}{1-i}\sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n} \\ &= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \end{aligned}$$

(13)

$$\begin{aligned} \frac{1}{4z-z^2} &= \frac{1}{z}\frac{1}{4-z} \\ &= \frac{1}{4z}\frac{1}{1-\frac{z}{4}} \\ &= \frac{1}{4z}\sum_{n=0}^{\infty} \frac{z^n}{4^n} \\ &= \frac{1}{4z}(1+\sum_{n=1}^{\infty} \frac{z^n}{4^n}) \end{aligned}$$

$$= \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^{n}}{4^{n+2}}$$