Solutions - Sections 60, 61, 62

(1)

$$\begin{aligned} f(z) &= z^2 \sin(\frac{1}{z^2}) \\ &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\frac{1}{z^2})^{2n+1} \\ &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n+2}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}} \end{aligned}$$

(3)

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \frac{1}{1+(1/z)} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} (\frac{-1}{z})^n \qquad \text{for } |z| > 1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n} \end{aligned}$$

(4) First, assume $0 < |z| < 1$.

$$\begin{aligned} f(z) &= \frac{1}{z^2} \frac{1}{1-z} \\ &= \frac{1}{z^2} \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1 \\ &= \sum_{n=0}^{\infty} z^{n-2} \\ &= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n \end{aligned}$$

Now, assume $|z| > 1$.

$$\begin{aligned} f(z) &= \frac{1}{z^2} \frac{1}{-z} \frac{1}{1-(1/z)} \\ &= \frac{-1}{z^3} \sum_{n=0}^{\infty} (\frac{1}{z})^n \qquad \text{for } |z| > 1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{-1}{z^{n+3}} \\
&= \sum_{n=3}^{\infty} \frac{-1}{z^{n}}
\end{aligned}
$$

(5a) Let $f(z) = \frac{z+1}{z-1}$. The Maclaurin series is valid for $|z| < 1$, which is the largest circle centered at $z = 0$ in which f is analytic.

$$
\begin{aligned}
f(z) &= \frac{z+1}{z-1} \\
&= (-z-1)\frac{1}{1-z} \\
&= (-z-1)\sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1 \\
&= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \\
&= -(\sum_{n=1}^{\infty} z^n) - 1 - (\sum_{n=1}^{\infty} z^n) \\
&= -1 - \sum_{n=1}^{\infty} 2z^n
\end{aligned}
$$

(5b) f is also analytic in the annulus $1 < |z| < \infty$.

$$
\begin{aligned}
f(z) &= (z+1)(\frac{1}{z})(\frac{1}{1-(1/z)}) \\
&= (1+\frac{1}{z})\sum_{n=0}^{\infty}(\frac{1}{z})^n \qquad \text{for } |z| > 1 \\
&= \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \\
&= 1 + \sum_{n=1}^{\infty} \frac{2}{z^n}
\end{aligned}
$$