Solutions - Sections 63 - 66

(3) We start by finding the Taylor series for $\frac{1}{z}$ centered at $z = 2$. Then differentiate both sides.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2 + (z-2)} \\
&= \frac{1}{2}\frac{1}{1-(-(z-2)/2)} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n \\
\frac{-1}{z^2} &= \sum_{n=1}^{\infty}\frac{(-1)^n n}{2^{n+1}}(z-2)^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)}{2^{n+2}}(z-2)^n \\
\frac{1}{z^2} &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n
\end{aligned}$$

(6) We can follow example 4 in section 59 to find that

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

for $|z-1| < 1$. Taking antiderivatives of both sides, we get

$$\log z = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(z-1)^{n+1} + C = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n + C$$

We can use any branch cut that doesn't intersect the circle $|z-1| < 1$, so we can choose the principal branch $-\pi < \theta < \pi$. For the principal branch, $\operatorname{Log} 1 = 0$. Plugging $z = 1$ into the right side of the series for $\log z$, all the terms in the sum are 0, so $C = 0$.

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n$$