

Solutions - Sections 69, 70, 71

(1a)

$$\begin{aligned} \frac{1}{z+z^2} &= \frac{1}{z} \frac{1}{1+z} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^n \\ &= \sum_{n=0}^{\infty} (-1)^n z^{n-1} \end{aligned}$$

The coefficient of z^{-1} is $(-1)^0 = 1$.

(1b)

$$\begin{aligned} z \cos\left(\frac{1}{z}\right) &= z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{2n}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{2n-1}} \end{aligned}$$

The coefficient of z^{-1} is $\frac{(-1)^1}{2!} = -\frac{1}{2}$.

(1c)

$$\begin{aligned} \frac{z - \sin z}{z} &= \frac{1}{z} \left(z - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \right) \\ &= \frac{1}{z} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} \end{aligned}$$

The terms in the series all have even powers, from z^2 up. There is no z^{-1} term, so the residue is 0.

(2a) Let C be the circle $|z| = 3$, positively oriented. The only singularity of $\frac{e^{-z}}{z^2}$ is at $z = 0$, which is inside C . We need $\text{Res}_0 f(z)$.

$$\begin{aligned} \frac{e^{-z}}{z^2} &= \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{n-2} \end{aligned}$$

The coefficient of z^{-1} is $\frac{(-1)^1}{1!} = -1$, so

$$\int_C \frac{e^{-z}}{z^2} dz = 2\pi i \text{Res}_0 f(z) = -2\pi i$$

(2b) The only singularity of $\frac{e^{-z}}{(z-1)^2}$ is at $z = 1$, which is inside C . We need $\text{Res}_1 f(z)$.

$$\begin{aligned} \frac{e^{-z}}{(z-1)^2} &= \frac{1}{(z-1)^2} e^{-(z-1)} e^{-1} \\ &= \frac{1}{e} \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-1)^n \\ &= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-1)^{n-2} \\ \text{Res}_1 f(z) &= \frac{1}{e} \frac{(-1)^1}{1!} = \frac{-1}{e} \\ \int_C \frac{e^{-z}}{(z-1)^2} dz &= 2\pi i \text{Res}_1 f(z) = \frac{-2\pi i}{e} \end{aligned}$$

(2c) The only singularity of $z^2 e^{\frac{1}{z}}$ is at $z = 0$, which is inside C . We need $\text{Res}_0 f(z)$.

$$\begin{aligned} z^2 e^{\frac{1}{z}} &= z^2 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^{n-2}} \\ \text{Res}_0 f(z) &= \frac{1}{3!} = \frac{1}{6} \\ \int_C z^2 e^{\frac{1}{z}} dz &= (2\pi i) \left(\frac{1}{6}\right) = \frac{\pi i}{3} \end{aligned}$$