Solutions - Sections 72, 73, 74

(2a) $z^{\frac{1}{4}}$ is analytic at $z=-1$, so

$$\operatorname{Res}_{-1}\frac{z^{\frac{1}{4}}}{z+1}=(-1)^{\frac{1}{4}}=(e^{i\pi})^{\frac{1}{4}}=e^{i\frac{\pi}{4}}=\frac{1+i}{\sqrt{2}}$$

(2b) $\frac{\operatorname{Log} z}{(z^2+1)^2}=\frac{\operatorname{Log} z}{(z+i)^2(z-i)^2}$. $\frac{\operatorname{Log} z}{(z+i)^2}$ is analytic at $z=i$. So

$$\begin{aligned}\operatorname{Res}_{i}\frac{\operatorname{Log} z}{(z^2+1)^2} &= \frac{d}{dz}\frac{\operatorname{Log} z}{(z+i)^2}\bigg|_{z=i}\\ &= \frac{\frac{1}{z}(z+i)^2-(\operatorname{Log} z)2(z+i)}{(z+i)^4}\bigg|_{z=i}\\ &= \frac{\frac{1}{i}(2i)^2-(i\frac{\pi}{2})2(2i)}{16}\\ &= \frac{4i+2\pi}{16}\\ &= \frac{2i+\pi}{8}\end{aligned}$$

(2c) $\frac{z^{\frac{1}{2}}}{(z^2+1)^2}=\frac{z^{\frac{1}{2}}}{(z+i)^2(z-i)^2}$. $\frac{z^{\frac{1}{2}}}{(z+i)^2}$ is analytic at $z=i$. So

$$\begin{aligned}\operatorname{Res}_{i}\frac{z^{\frac{1}{2}}}{(z^2+1)^2} &= \frac{d}{dz}\frac{z^{\frac{1}{2}}}{(z+i)^2}\bigg|_{z=i}\\ &= \frac{\frac{1}{2}z^{-\frac{1}{2}}(z+i)^2-(z^{\frac{1}{2}})2(z+i)}{(z+i)^4}\bigg|_{z=i}\\ &= \frac{\frac{1}{2}e^{-i\frac{\pi}{4}}(2i)^2-(e^{\frac{\pi}{4}})2(2i)}{16}\\ &= \frac{\frac{1}{2}\frac{1-i}{\sqrt{2}}(-4)-\frac{1+i}{\sqrt{2}}4i}{16}\\ &= \frac{(-1+i)\sqrt{2}+(-i+1)2\sqrt{2}}{16}\\ &= \frac{(1-i)\sqrt{2}}{16}\end{aligned}$$

(3) $f(z)=\frac{3z^3+2}{(z-1)(z^2+9)}=\frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$ has three simple poles, at 1, $3i$ and $-3i$.

$$\begin{aligned}\operatorname{Res}_{1}f(z) &= \frac{3(1)^3+2}{(1+3i)(1-3i)}\\ &= \frac{5}{1+9}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \\
\text{Res}_{3i} f(z) &= \frac{3(3i)^3+2}{(3i-1)(3i+3i)} \\
&= \frac{3(-27i)+2}{-18-6i} \\
&= \frac{-81i+2}{-18-6i} \cdot \frac{-18+6i}{-18+6i} \\
&= \frac{450+1470i}{360} \\
&= \frac{15+49i}{12} \\
\text{Res}_{-3i} f(z) &= \frac{3(-3i)^3+2}{(-3i-1)(-3i-3i)} \\
&= \frac{3(27i)+2}{-18+6i} \\
&= \frac{81i+2}{-18+6i} \cdot \frac{-18-6i}{-18-6i} \\
&= \frac{450-1470i}{360} \\
&= \frac{15-49i}{12}
\end{aligned}
$$

(a) The contour C_a, $|z-2| = 2$ only contains the pole at $z = 1$, so

$$\int_{C_a} f(z)\,dz = 2\pi i \text{Res}_1 f(z) = 2\pi i \frac{1}{2} = \pi i$$

(b) The contour C_b, $|z| = 4$ contains all 3 poles.

$$
\begin{aligned}
\int_{C_b} f(z)\,dz &= 2\pi i(\text{Res}_1 f(z) + \text{Res}_{3i} f(z) + \text{Res}_{-3i} f(z)) \\
&= 2\pi i(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12}) \\
&= 2\pi i(\frac{1}{2} + \frac{30}{12}) \\
&= 2\pi i(3) = 6\pi i
\end{aligned}
$$

(4) $\frac{1}{z^3(z+4)}$ has a pole order 3 at $z = 0$ and a simple pole at $z = -4$.

$$
\begin{aligned}
\frac{d}{dz}(\frac{1}{z+4}) &= \frac{-1}{(z+4)^2} \\
\frac{d^2}{dz^2}(\frac{1}{z+4}) &= \frac{2}{(z+4)^3} \\
\text{Res}_0 f(z) &= \frac{1}{2!}\frac{2}{(0+4)^3}
\end{aligned}
$$

$$\begin{aligned} &= \frac{1}{64} \\ \text{Res}_{-4} f(z) &= \frac{1}{(-4)^3} \\ &= \frac{-1}{64} \end{aligned}$$

(a) The contour C_a, $|z| = 2$ only contains the pole at 0.

$$\int_{C_a} f(z)\, dz = 2\pi i \text{Res}_{0} f(z) = 2\pi i \frac{1}{64} = \frac{\pi i}{32}$$

(b) The contour C_b, $|z + 2| = 3$ contains both poles.

$$\int_{C_a} f(z)\, dz = 2\pi i(\text{Res}_{0} f(z) + \text{Res}_{-4} f(z)) = 2\pi i(\frac{1}{64} - \frac{1}{64}) = 0$$