Solutions - Sections 75, 76

(3a) $z \sec z = \frac{z}{\cos z}$. $\cos z = 0$ at $z_n = \frac{\pi}{2} + n\pi$ and its derivative, $-\sin z_n = (-1)^{n+1}$, is non-zero at those points.

$$\text{Res}_{\,z_n} = \frac{z_n}{-\sin z_n} = (-1)^{n+1} z_n$$

(4a) $\tan z = \frac{\sin z}{\cos z}$. $\cos z = 0$ at $z_n = \frac{\pi}{2} + n\pi$ and its derivative, $-\sin z_n = (-1)^{n+1}$, is non-zero at those points. Of those poles, only $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ are inside C, the circle $|z| = 2$ positively oriented.

$$\begin{aligned}
\text{Res}_{\,\frac{\pi}{2}} \tan z &= \frac{\sin \frac{\pi}{2}}{-\sin \frac{\pi}{2}} = -1 \\
\text{Res}_{\,\frac{-\pi}{2}} \tan z &= \frac{\sin \frac{-\pi}{2}}{-\sin \frac{-\pi}{2}} = -1 \\
\int_C \tan z \, dz &= 2\pi i(-1-1) = -4\pi i
\end{aligned}$$

(7) Assume q is analytic at z_0, $q(z_0) = 0$ and $q'(z_0) \neq 0$. q has a zero of order 1 at z_0, so $q(z) = (z - z_0)g(z)$ where g is analytic at z_0 and $g(z_0) \neq 0$. Then

$$f(z) = \frac{1}{[q(z)]^2} = \frac{1}{(z-z_0)^2 [g(z)]^2} = \frac{\phi(z)}{(z-z_0)^2}$$

where $\phi(z) = \frac{1}{[g(z)]^2}$ is analytic and non-zero at z_0.

$$\begin{aligned}
\text{Res}_{\,z_0} f(z) &= \frac{\phi'(z_0)}{1!} \\
&= \frac{-2g'(z_0)}{[g(z_0)]^3} \\
q'(z) &= g(z) + (z - z_0)g'(z) \\
q'(z_0) &= g(z_0) \\
q''(z) &= 2g'(z) + (z - z_0)g''(z) \\
q''(z_0) &= 2g'(z_0) \\
\text{Res}_{\,z_0} f(z) &= \frac{-q''(z_0)}{[q'(z_0)]^3}
\end{aligned}$$