Problems - Practice Integrals

Evaluate the following integrals.

(1)

$$\int_0^\infty \frac{dx}{x^2+4}$$

(No need to simplify the answer.)

$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$ has simple poles at $\pm 2i$. Only $2i$ is inside C.

$$\begin{aligned}
\int_C \frac{1}{z^2+4}dz &= 2\pi i \text{Res}_{\,2i}\frac{1}{z^2+4} \\
&= 2\pi i \frac{1}{2i+2i} \\
&= \frac{\pi}{2} \\
\int_C \frac{1}{z^2+4}dz &= \int_0^R \frac{1}{x^2+4}dx + \int_{C_-} \frac{1}{z^2+4}dz + \int_{C_R} \frac{1}{z^2+4}dz \\
\left|\int_{C_R} \frac{1}{z^2+4}dz\right| &\le \frac{1}{R^2-4}(\pi R) \\
\int_{C_-} \frac{1}{z^2+4}dz &= -\int_0^R \frac{1}{(-t)^2+4}(-1)dt \\
&= \int_0^R \frac{1}{t^2+4}dt
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{\pi R}{R^2-4} = 0$, we have that

$$\begin{aligned}
2\int_0^\infty \frac{dx}{x^2+4} &= \frac{\pi}{2} \\
\int_0^\infty \frac{dx}{x^2+4} &= \frac{\pi}{4}
\end{aligned}$$

(2)

$$\int_0^\infty \frac{dx}{x^3+1}$$

(Come back along angle $\frac{2\pi}{3}$).

$\frac{1}{z^3+1}$ has simple poles at $e^{\frac{i\pi}{3}}, e^{\frac{i3\pi}{3}}, e^{\frac{i5\pi}{3}}$. Only $e^{\frac{i\pi}{3}}$ is inside C.

$$\begin{aligned}
\int_C \frac{1}{z^3+1}dz &= 2\pi i \text{Res}_{\,e^{\frac{i\pi}{3}}}\frac{1}{z^3+1} \\
&= 2\pi i \frac{1}{3(e^{\frac{i\pi}{3}})^2}
\end{aligned}$$

$$\begin{aligned}
\int_C \frac{1}{z^3+1}dz &= \int_0^R \frac{1}{x^3+1}dx + \int_{C_-} \frac{1}{z^3+1}dz + \int_{C_R} \frac{1}{z^3+1}dz \\
\left|\int_{C_R} \frac{1}{z^3+1}dz\right| &\leq \frac{1}{R^3-1}(\frac{2\pi R}{3}) \\
\int_{C_-} \frac{1}{z^3+1}dz &= -\int_0^R \frac{1}{(e^{\frac{i2\pi}{3}}t)^3+1}(e^{\frac{i2\pi}{3}})dt \\
&= -e^{\frac{i2\pi}{3}} \int_0^R \frac{1}{t^3+1}dt
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{2\pi R}{3(R^3-1)} = 0$, we have that

$$\begin{aligned}
(1-e^{\frac{i2\pi}{3}}) \int_0^\infty \frac{dx}{x^3+1} &= 2\pi i \frac{1}{3e^{\frac{i2\pi}{3}}} \\
\int_0^\infty \frac{dx}{x^2+1} &= 2\pi i \frac{1}{3e^{\frac{i2\pi}{3}}(1-e^{\frac{i2\pi}{3}})}
\end{aligned}$$

(3)

$$\int_0^\infty \frac{x\,dx}{x^3+1}$$

(Come back along angle $\frac{2\pi}{3}$).

$\frac{z}{z^3+1}$ has simple poles at $e^{\frac{i\pi}{3}}, e^{\frac{i3\pi}{3}}, e^{\frac{i5\pi}{3}}$. Only $e^{\frac{i\pi}{3}}$ is inside C.

$$\begin{aligned}
\int_C \frac{z}{z^3+1}dz &= 2\pi i \text{Res}_{e^{\frac{i\pi}{3}}} \frac{z}{z^3+1} \\
&= 2\pi i \frac{1}{3e^{\frac{i\pi}{3}}} \\
\int_C \frac{z}{z^3+1}dz &= \int_0^R \frac{x}{x^3+1}dx + \int_{C_-} \frac{z}{z^3+1}dz + \int_{C_R} \frac{z}{z^3+1}dz \\
\left|\int_{C_R} \frac{z}{z^3+1}dz\right| &\leq \frac{R}{R^3-1}(\frac{2\pi R}{3}) \\
\int_{C_-} \frac{z}{z^3+1}dz &= -\int_0^R \frac{e^{\frac{i2\pi}{3}}t}{(e^{\frac{i2\pi}{3}}t)^3+1}(e^{\frac{i2\pi}{3}})dt \\
&= -e^{\frac{i4\pi}{3}} \int_0^R \frac{t}{t^3+1}dt
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{2\pi R^2}{3(R^3-1)} = 0$, we have that

$$\begin{aligned}
(1-e^{\frac{i4\pi}{3}}) \int_0^\infty \frac{xdx}{x^3+1} &= 2\pi i \frac{1}{3e^{\frac{i\pi}{3}}} \\
\int_0^\infty \frac{xdx}{x^3+1} &= 2\pi i \frac{1}{3e^{\frac{i\pi}{3}}(1-e^{\frac{i4\pi}{3}})}
\end{aligned}$$