Solutions - Sections 78, 79

In all solutions, C_R is the upper half circle with radius R and center 0 (R is very large). C is the contour from $-R$ to R along the real axis then back along C_R.

(1) $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ has simple poles at $\pm i$. Only i is inside C.

$$\begin{aligned}
\int_C \frac{1}{z^2+1}dz &= 2\pi i \text{Res}_i \frac{1}{z^2+1} \\
&= 2\pi i \frac{1}{i+i} \\
&= \pi \\
\int_C \frac{1}{z^2+1}dz &= \int_0^R \frac{1}{x^2+1}dx + \int_{C_-} \frac{1}{z^2+1}dz + \int_{C_R} \frac{1}{z^2+1}dz \\
\left|\int_{C_R} \frac{1}{z^2+1}dz\right| &\leq \frac{R}{R^2-1}(\pi R) \\
\int_{C_-} \frac{1}{z^2+1}dz &= -\int_0^R \frac{1}{(-t)^2+1}(-1)dt \\
&= \int_0^R \frac{1}{t^2+1}dt
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{\pi R}{R^2-1} = 0$, we have that

$$\begin{aligned}
2\int_0^\infty \frac{dx}{x^2+1} &= \pi \\
\int_0^\infty \frac{dx}{x^2+1} &= \frac{\pi}{2}
\end{aligned}$$

(2) $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$ has poles of order 2 at $\pm i$. Only i is inside C.

$$\begin{aligned}
\text{Res}_i \frac{1}{(z+i)^2(z-i)^2} &= \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right|_{z=i} \\
&= \frac{-2}{(i+i)^3} = \frac{1}{4i} \\
\int_C \frac{1}{(z^2+1)^2}dz &= 2\pi i \text{Res}_i \frac{1}{(z^2+1)^2} \\
&= \frac{\pi}{2} \\
\int_C \frac{1}{(z^2+1)^2}dz &= \int_0^R \frac{1}{(x^2+1)^2}dx + \int_{C_-} \frac{1}{(z^2+1)^2}dz + \int_{C_R} \frac{1}{(z^2+1)^2}dz \\
\left|\int_{C_R} \frac{1}{(z^2+1)^2}dz\right| &\leq \frac{1}{R^4-2R^2-1}(\pi R) \\
\int_{C_-} \frac{1}{(z^2+1)^2}dz &= -\int_0^R \frac{1}{((-t)^2+1)^2}(-1)dt \\
&= \int_0^R \frac{1}{(t^2+1)^2}dt
\end{aligned}$$

Since $\lim_{R\to\infty} \frac{\pi R}{R^4-2R^2-1} = 0$, we have that

$$\begin{aligned} 2\int_0^\infty \frac{dx}{(x^2+1)^2} &= \frac{\pi}{2} \\ \int_0^\infty \frac{dx}{(x^2+1)^2} &= \frac{\pi}{4} \end{aligned}$$

(3) Since $z^4 = -1$ has solutions $e^{i(\frac{\pi}{4}+\frac{\pi k}{2})}$, $\frac{1}{z^4+1}$ has simple poles at $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$. Only $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ is inside C.

$$\begin{aligned} \text{Res}_{e^{\frac{i\pi}{4}}} \frac{1}{z^4+1} &= \left.\frac{1}{4z^3}\right|_{e^{\frac{i\pi}{4}}} \\ &= \frac{1}{4e^{\frac{i3\pi}{4}}} = \frac{e^{-\frac{i3\pi}{4}}}{4} \\ \text{Res}_{e^{\frac{i3\pi}{4}}} \frac{1}{z^4+1} &= \left.\frac{1}{4z^3}\right|_{e^{\frac{i3\pi}{4}}} \\ &= \frac{1}{4e^{\frac{i9\pi}{4}}} = \frac{e^{-\frac{i\pi}{4}}}{4} \\ \int_C \frac{1}{z^4+1}dz &= 2\pi i \left(\frac{e^{-\frac{i3\pi}{4}}}{4} + \frac{e^{-\frac{i\pi}{4}}}{4}\right) \\ &= \frac{\pi i}{2}\left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) \\ &= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) \\ &= \frac{\pi}{\sqrt{2}} \\ \int_C \frac{1}{z^4+1}dz &= \int_0^R \frac{1}{x^4+1}dx + \int_{C_-} \frac{1}{z^4+1}dz + \int_{C_R} \frac{1}{z^4+1}dz \\ \left|\int_{C_R} \frac{1}{z^4+1}dz\right| &\le \frac{1}{R^4-1}\pi R \\ \int_{C_-} \frac{1}{z^4+1}dz &= -\int_0^R \frac{1}{(-t)^4+1}(-1)dt \\ &= \int_0^R \frac{1}{t^4+1}dt \end{aligned}$$

Since $\lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0$, we have that

$$\begin{aligned} 2\int_0^\infty \frac{dx}{x^4+1} &= \frac{\pi}{\sqrt{2}} \\ \int_0^\infty \frac{dx}{x^4+1} &= \frac{\pi}{2\sqrt{2}} \end{aligned}$$

(4) $\frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$ has simple poles at $\pm i, \pm 2i$. Only $i, 2i$ are inside C.

$$\begin{aligned}
\text{Res}_{\,i}\frac{z^2}{(z^2+1)(z^2+4)} &= \left.\frac{z^2}{2z(z^2+4)+(z^2+1)(2z)}\right|_i \\
&= \frac{-1}{2i(-1+4)+0} = \frac{-1}{6i} \\
\text{Res}_{\,2i}\frac{z^2}{(z^2+1)(z^2+4)} &= \left.\frac{z^2}{2z(z^2+4)+(z^2+1)(2z)}\right|_{2i} \\
&= \frac{-4}{0+(-4+1)(4i)} = \frac{1}{3i} \\
\int_C \frac{z^2}{(z^2+1)(z^2+4)}dz &= 2\pi i\left(\frac{-1}{6i}+\frac{1}{3i}\right) \\
&= 2\pi i\left(\frac{1}{6i}\right) \\
&= \frac{\pi}{3} \\
\int_C \frac{z^2}{(z^2+1)(z^2+4)}dz &= \int_0^R \frac{x^2}{(x^2+1)(x^2+4)}dx + \int_{C_-} + \int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}dz \\
\left|\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}dz\right| &= \left|\int_{C_R}\frac{z^2}{z^4+5z^2+4}dz\right| \\
&\leq \frac{R^2}{R^4-5R^2-4}(\pi R) \\
\int_{C_-}\frac{z^2}{(z^2+1)(z^2+4)}dz &= -\int_0^R \frac{(-t)^2}{((-t)^2+1)((-t)^2+4)}(-1)dt \\
&= \int_0^R \frac{t^2}{(t^2+1)(t^2+4)}dt
\end{aligned}$$

Since $\lim_{R\to\infty}\frac{\pi R^3}{R^4-5R^2-4} = 0$, we have that

$$\begin{aligned}
2\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)} &= \frac{\pi}{3} \\
\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)} &= \frac{\pi}{6}
\end{aligned}$$