Solutions - 81 - Improper Integrals with Jordan's Lemma

In all solutions, C_R is the upper half circle with radius R and center 0 (R is very large). C is the contour from $-R$ to R along the real axis then back along C_R.

(4) Evaluate $\int_0^\infty \frac{x\sin 2x}{x^2+3}dx$. In order to do this, we first evaluate

$$\int_C \frac{ze^{i2z}}{z^2+3}dz = \int_0^R \frac{xe^{i2x}}{x^2+3}dx + \int_{C_-} \frac{ze^{i2z}}{z^2+3}dz + \int_{C_R} \frac{ze^{i2z}}{z^2+3}dz$$

$\frac{ze^{i2z}}{z^2+3}$ has simple poles at $\pm i\sqrt{3}$, of which only $i\sqrt{3}$ is inside C.

$$\begin{aligned}
\text{Res}_{\,i\sqrt{3}} \frac{ze^{i2z}}{z^2+3} &= \left.\frac{ze^{i2z}}{2z}\right|_{z=i\sqrt{3}} \\
&= \frac{e^{-2\sqrt{3}}}{2} \\
\int_C \frac{ze^{i2z}}{z^2+3}dz &= 2\pi i(\frac{e^{-2\sqrt{3}}}{2}) \\
&= i\pi e^{-2\sqrt{3}} \\
\int_{C_-} \frac{ze^{i2z}}{z^2+3}dz &= -\int_0^R \frac{-te^{-i2t}}{(-t)^2+3}(-1)dt \\
&= -\int_0^R \frac{te^{-i2t}}{t^2+3}dt
\end{aligned}$$

On C_R, $\left|\frac{z}{z^2+3}\right| \le \frac{R}{R^2-3}$ and $\lim_{R\to\infty} \frac{R}{R^2-3} = 0$ so by Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{i2z}}{z^2+3}dz = 0$$

Now we put it all together.

$$\begin{aligned}
i\pi e^{-2\sqrt{3}} &= \int_0^\infty \frac{xe^{i2x}}{x^2+3}dx - \int_0^\infty \frac{xe^{-i2x}}{x^2+3}dx \\
&= 2i\int_0^\infty \frac{x\sin 2x}{x^2+3}dx \\
\int_0^\infty \frac{x\sin 2x}{x^2+3}dx &= \frac{\pi e^{-2\sqrt{3}}}{2}
\end{aligned}$$

(6) Evaluate $\int_0^\infty \frac{x^3\sin ax}{x^4+4}dx$, where $a>0$. In order to do this, we first evaluate

$$\int_C \frac{z^3e^{iaz}}{z^4+4}dz = \int_0^R \frac{x^3e^{iax}}{x^4+4}dx + \int_{C_-} \frac{z^3e^{iaz}}{z^4+4}dz + \int_{C_R} \frac{z^3e^{iaz}}{z^4+4}dz$$

$\frac{z^3 e^{iaz}}{z^4+4}$ has simple poles at $\pm 1 \pm i$, of which $1+i$ and $-1+i$ are inside C.

$$\begin{aligned}
\text{Res}_{\,1+i} \frac{z^3 e^{iaz}}{z^4+4} &= \left.\frac{z^3 e^{iaz}}{4z^3}\right|_{z=1+i} \\
&= \frac{e^{ia(1+i)}}{4} = \frac{e^{ia}e^{-a}}{4} \\
\text{Res}_{\,-1+i} \frac{z^3 e^{iaz}}{z^4+4} &= \left.\frac{z^3 e^{iaz}}{4z^3}\right|_{z=-1+i} \\
&= \frac{e^{ia(-1+i)}}{4} = \frac{e^{-ia}e^{-a}}{4} \\
\int_C \frac{z^3 e^{iaz}}{z^4+4} dz &= 2\pi i \left(\frac{e^{ia}e^{-a}}{4} + \frac{e^{-ia}e^{-a}}{4}\right) \\
&= \pi i \left(\frac{e^{ia}e^{-a}}{2} + \frac{e^{-ia}e^{-a}}{2}\right) \\
&= \frac{\pi i e^{-a}}{2}(e^{ia} + e^{-ia}) \\
&= \frac{\pi i e^{-a}}{2}(2\cos a) \\
&= i\pi e^{-a}\cos a \\
\int_{C_-} \frac{z^3 e^{iaz}}{z^4+4} dz &= -\int_0^R \frac{(-t)^3 e^{-iat}}{(-t)^4+4}(-1)dt \\
&= -\int_0^R \frac{t^3 e^{-iat}}{t^4+4} dt
\end{aligned}$$

On C_R, $\left|\frac{z^3}{z^4+4}\right| \leq \frac{R^3}{R^4-4}$ and $\lim_{R\to\infty} \frac{R^3}{R^4-4} = 0$ so by Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} \frac{z^3 e^{iaz}}{z^4+4} dz = 0$$

Now we put it all together.

$$\begin{aligned}
i\pi e^{-a}\cos a &= \int_0^\infty \frac{x^3 e^{iax}}{x^4+4} dx - \int_0^\infty \frac{x^3 e^{-iax}}{x^4+4} dx \\
&= 2i \int_0^\infty \frac{x^3 \sin ax}{x^4+4} dx \\
\int_0^\infty \frac{x^3 \sin ax}{x^4+4} dx &= \frac{\pi e^{-a}\cos a}{2}
\end{aligned}$$