Solutions - Section 81

In all solutions, C_R is the upper half circle with radius R and center 0 (R is very large). C is the contour from $-R$ to R along the real axis then back along C_R.

(2) Evaluate $\int_0^\infty \frac{\cos ax}{x^2+1}dx$ $(a > 0)$. In order to do this, we first evaluate

$$\int_C \frac{e^{iaz}}{z^2+1}dz = \int_0^R \frac{e^{iax}}{x^2+1}dx + \int_{C_-} \frac{e^{iaz}}{z^2+1}dz + \int_{C_R} \frac{e^{iaz}}{z^2+1}dz$$

$\frac{e^{iaz}}{z^2+1}$ has simple poles at $\pm i$, of which only i is inside C.

$$\begin{aligned}
\operatorname{Res}_i \frac{e^{iaz}}{z^2+1} &= \left.\frac{e^{iaz}}{2z}\right|_{z=i} \\
&= \frac{e^{-a}}{2i} \\
\int_C \frac{e^{iaz}}{z^2+1}dz &= 2\pi i(\frac{e^{-a}}{2i}) \\
&= \pi e^{-a} \\
\int_{C_-} \frac{e^{iaz}}{z^2+1}dz &= -\int_0^R \frac{e^{ia(-t)}}{(-t)^2+1}(-1)dt \\
&= \int_0^R \frac{e^{-iat}}{t^2+1}dt
\end{aligned}$$

On C_R, $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{R^2-1}$ and $\lim_{R\to\infty} \frac{1}{R^2-1} = 0$ so by Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z^2+1}dz = 0$$

Now we put it all together, using the fact that the original integrand is even.

$$\begin{aligned}
\pi e^{-a} &= \int_0^\infty \frac{e^{iax}}{x^2+1}dx + \int_0^\infty \frac{e^{-iax}}{x^2+1}dx \\
&= \int_0^\infty \frac{2\cos ax}{x^2+1}dx \\
\int_0^\infty \frac{\cos ax}{x^2+1}dx &= \frac{\pi e^{-a}}{2}
\end{aligned}$$

(9) Evaluate $\lim_{R\to\infty} \int_{-R}^R \frac{\sin x}{x^2+4x+5}dx$. In order to do this, we first evaluate

$$\int_C \frac{e^{iz}}{z^2+4z+5}dz = \int_{-R}^R \frac{e^{ix}}{x^2+4x+5}dx + \int_{C_R} \frac{e^{iz}}{z^2+4z+5}dz$$

$\frac{e^{iz}}{z^2+4z+5}$ has simple poles at $-2 \pm i$, of which only $-2+i$ is inside C.

$$\begin{aligned}
\text{Res}_{-2+i} \frac{e^{iz}}{z^2+4z+5} &= \left.\frac{e^{iz}}{2z+4}\right|_{z=-2+i} \\
&= \frac{e^{-1-2i}}{2i} \\
\int_C \frac{e^{iz}}{z^2+4z+5} dz &= 2\pi i (\frac{e^{-1-2i}}{2i}) \\
&= \pi e^{-1} e^{-2i} \\
&= \frac{\pi}{e}(\cos 2 - i \sin 2)
\end{aligned}$$

On C_R, $\left|\frac{1}{z^2+4z+5}\right| \leq \frac{1}{R^2-4R-5}$ and $\lim_{R\to\infty} \frac{1}{R^2-4R-5} = 0$ so by Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2+4z+5} dz = 0$$

Now we put it all together, matching imaginary parts.

$$\begin{aligned}
\frac{\pi}{e}(\cos 2 - i \sin 2) &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+4x+5} dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x + i \sin x}{x^2+4x+5} dx \\
\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x^2+4x+5} dx &= \frac{\pi}{e}(-\sin 2)
\end{aligned}$$