(4) Let \( f(z) = \frac{(\log z)^2}{z^2 + 1} \) with the branch \(-\frac{\pi}{2} < \theta < \frac{3\pi}{2}\). We will use the same path \( C \) as in problem (1) above. The only singularity of \( f \) inside \( C \) is at \( z = i \) (the other singularity at \(-i\) is not only outside \( C \), it is on the branch cut). It is a simple pole, so

\[
\text{Res}_i f(z) = \frac{p(i)}{q'(i)} = \frac{(\log i)^2}{2i} = \frac{(i\pi^2)^2}{8i} = \frac{i\pi^2}{8}
\]

\[
2\pi i \text{Res}_i f(z) = -\frac{\pi^3}{4}
\]

\[
\int_C f(z) \, dz = \int_{-\rho}^{\rho} f(z) \, dz + \int_{C_\rho} f(z) \, dz + \int_{\rho}^{R} f(z) \, dz + \int_{C_R} f(z) \, dz
\]

First, \( C_\rho \).

\[
|f(z)| = \frac{|\ln r + i\theta|^2}{|z^2 + 1|} \leq \frac{\ln^2 \rho + \pi^2}{1 - \rho^2}
\]

\[
\left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{\pi \rho (\ln^2 \rho + \pi^2 \rho)}{1 - \rho^2}
\]

\[
\lim_{\rho \to 0} \left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{\pi 0 + \pi^3 0}{1 - 0} = 0
\]

Second, \( C_R \).

\[
|f(z)| = \frac{|\ln r + i\theta|^2}{|z^2 + 1|} \leq \frac{\ln^2 R + \pi^2}{R^2 - 1}
\]

\[
\left| \int_{C_R} f(z) \, dz \right| \leq \frac{\pi R (\ln^2 R + \pi^2 R)}{R^2 - 1}
\]

\[
\lim_{R \to \infty} \left| \int_{C_R} f(z) \, dz \right| = 0
\]
Third, the negative real axis piece.

\[
\int_{-\rho}^{-R} f(z)dz = \int_{R}^{\rho} \frac{(\log(-x))^2}{(-x)^2 + 1}(-dx)
\]
\[
= \int_{\rho}^{R} \frac{(\ln x + i\pi)^2}{x^2 + 1}dx
\]
\[
= \int_{\rho}^{R} \frac{(\ln x)^2 - \pi^2 + i2\pi \ln x}{x^2 + 1}dx
\]
\[
= \frac{\int_{\rho}^{R} (\ln x)^2}{x^2 + 1}dx - \pi^2 \int_{\rho}^{R} \frac{1}{x^2 + 1}dx + i2\pi \int_{\rho}^{R} \frac{\ln x}{x^2 + 1}dx
\]

Note that from exercise (1) in section 79, \( \int_{0}^{\infty} \frac{1}{x^2 + 1}dx = \frac{\pi}{2} \).

Now, put everything together and take the limits \( \rho \to 0 \) and \( R \to \infty \).

\[
\int_{-\rho}^{-R} f(z)dz + \int_{C_R} f(z)dz + \int_{\rho}^{R} f(z)dz + \int_{C_{\rho}} f(z)dz = -\frac{\pi^3}{4}
\]
\[
2 \int_{0}^{\infty} \frac{(\ln x)^2}{x^2 + 1}dx - \pi^2 \int_{0}^{\infty} \frac{1}{x^2 + 1}dx + i2\pi \int_{0}^{\infty} \frac{\ln x}{x^2 + 1}dx = \frac{\pi^3}{4}
\]
\[
2 \int_{0}^{\infty} \frac{(\ln x)^2}{x^2 + 1}dx - (\pi^2) \frac{\pi}{2} + i2\pi \int_{0}^{\infty} \frac{\ln x}{x^2 + 1}dx = -\frac{\pi^3}{4}
\]
\[
2 \int_{0}^{\infty} \frac{(\ln x)^2}{x^2 + 1}dx + i2\pi \int_{0}^{\infty} \frac{\ln x}{x^2 + 1}dx = \frac{\pi^3}{4}
\]
\[
\int_{0}^{\infty} \frac{(\ln x)^2}{x^2 + 1}dx + i\pi \int_{0}^{\infty} \frac{\ln x}{x^2 + 1}dx = \frac{\pi^3}{4}
\]

Matching real parts and imaginary parts,

\[
\int_{0}^{\infty} \frac{(\ln x)^2}{x^2 + 1}dx = \frac{\pi^3}{8}, \quad \int_{0}^{\infty} \frac{\ln x}{x^2 + 1}dx = 0
\]

(6a) Let \( f(z) = \frac{z^{-\frac{1}{2}}}{z^2 + 1} \) with branch \(-\frac{n}{2} < \theta < \frac{3\pi}{2}\). We will use the same path \( C \) as in problem (1) above. The only singularity of \( f \) inside \( C \) is at \( z = i \) (the other singularity at \(-i\) is not only outside \( C \), it is on the branch cut). It is a simple pole, so

\[
\text{Res}_i f(z) = \frac{p(i)}{q'(i)} = \frac{e^{-\frac{1}{2} \log i}}{-\frac{2i}{2i}} = \frac{e^{-\frac{i \pi}{2}}}{2i} = \frac{e^{-\frac{\pi}{4}}}{2i}
\]
\[ 2\pi i \text{Res}_i f(z) = \pi e^{-i\pi} \]

\[ \int_C f(z) \, dz = \int_{-\rho}^{-R} f(z) \, dz + \int_{C_\rho} f(z) \, dz + \int_{\rho}^{R} f(z) \, dz + \int_{C_R} f(z) \, dz \]

\[ = \pi e^{-i\pi/4} \]

First, \( C_\rho \).

\[ |f(z)| = \frac{|z|^{-\frac{1}{2}}}{|z^2 + 1|} \]

\[ \leq \frac{\rho^{-\frac{1}{2}}}{1 - \rho^2} \]

\[ \left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{\pi \rho^2}{1 - \rho^2} \]

\[ \lim_{\rho \to 0} \left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{0}{1 - 0} = 0 \]

Second, \( C_R \).

\[ |f(z)| = \frac{|z|^{-\frac{1}{2}}}{|z^2 + 1|} \]

\[ \leq \frac{R^{-\frac{1}{2}}}{R^2 - 1} \]

\[ \left| \int_{C_R} f(z) \, dz \right| \leq \frac{\pi R^2}{R^2 - 1} \]

\[ \lim_{R \to \infty} \left| \int_{C_R} f(z) \, dz \right| = 0 \]

Third, the negative real axis piece.

\[ \int_{-\rho}^{-R} f(z) \, dz = \int_{\rho}^{R} e^{-\frac{1}{2} \log(-x)} (-dx) \]

\[ = \int_{\rho}^{R} \frac{e^{-\frac{1}{2} \ln x + i\pi i}}{x^2 + 1} \, dx \]

\[ = \int_{\rho}^{R} \frac{e^{-\frac{1}{2} \ln x} (-i)}{x^2 + 1} \, dx \]

\[ = \int_{\rho}^{R} \frac{-ix^{-\frac{1}{2}}}{x^2 + 1} \, dx \]

Now, put everything together and take the limits \( \rho \to 0 \) and \( R \to \infty \).

\[ \int_{-\rho}^{-R} f(z) \, dz + \int_{C_\rho} f(z) \, dz + \int_{\rho}^{R} f(z) \, dz + \int_{C_R} f(z) \, dz = \pi e^{-i\pi/4} \]
\[
(1 - i) \int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2 + 1} \, dx + 0 + 0 = \frac{1 - i}{\sqrt{2}}
\]
\[
\int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2 + 1} \, dx = \frac{\pi}{\sqrt{2}}
\]

(6b) Let \( f(z) = \frac{z^{-\frac{1}{2}}}{z^2 + 1} \) with branch \( 0 < \theta < 2\pi \). Let \( C \) be the contour which goes counterclockwise around the circle \( C_R \) of radius \( R \) from \( \theta = 0 \) to \( 2\pi \), along the underside of the branch cut from \( R \) to \( \rho \), clockwise around the circle \( C_\rho \) of radius \( \rho \) from \( \theta = 2\pi \) down to \( 0 \), then along the top of the branch cut from \( \rho \) up to \( R \).

\( f(z) \) has two simple poles inside \( C \) at \( \pm i \).

\[
\text{Res}_{i} f(z) = \frac{p(i)}{q'(i)} = \frac{e^{-\frac{1}{2} \log i}}{2i} = \frac{e^{-\frac{i\pi}{4}}}{2i}
\]

\[
\text{Res}_{-i} f(z) = \frac{p(-i)}{q'(-i)} = \frac{e^{-\frac{1}{2} \log (-i)}}{-2i} = \frac{e^{-\frac{3i\pi}{4}}}{-2i}
\]

\[
2\pi i (\text{Res}_{i} f(z) + \text{Res}_{-i} f(z)) = \pi (e^{-\frac{i\pi}{4}} - e^{-\frac{3i\pi}{4}})
\]

\[
= \pi \left( \frac{(1 - i) - (-1 - i)}{\sqrt{2}} \right)
\]

\[
= \pi \sqrt{2}
\]

\[
\int_{C} f(z) \, dz = \int_{C_R} f(z) \, dz + \int_{R} f(z) \, dz + \int_{C_\rho} f(z) \, dz + \int_{\rho} f(z) \, dz
\]

\[
= \pi \sqrt{2}
\]

First, \( C_\rho \).

\[
|f(z)| = \frac{|z|^{-\frac{1}{2}}}{|z^2 + 1|}
\]

\[
\leq \frac{\rho^{-\frac{1}{2}}}{1 - \rho^2}
\]

\[
\left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{2\pi \rho^{\frac{3}{2}}}{1 - \rho^2}
\]

\[
\lim_{\rho \to 0} \left| \int_{C_\rho} f(z) \, dz \right| \leq \frac{0}{1 - 0} = 0
\]

Second, \( C_R \).

\[
|f(z)| = \frac{|z|^{-\frac{1}{2}}}{|z^2 + 1|}
\]

\[
\leq \frac{R^{-\frac{1}{2}}}{R^2 - 1}
\]

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\[
\left| \int_{C_R} f(z)\,dz \right| \leq \frac{2\pi R^\frac{1}{2}}{R^2 - 1}
\]

\[
\lim_{R \to \infty} \left| \int_{C_R} f(z)\,dz \right| = 0
\]

Third, the piece under the branch cut.
\[
\int_R^\rho f(z)\,dz = \int_R^\rho \frac{e^{-\frac{1}{2}(\ln x + i2\pi)}}{x^2 + 1}\,dx
\]
\[
= -\int_\rho^R \frac{e^{-\ln x + i\pi}}{x^2 + 1}\,dx
\]
\[
= \int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2 + 1}\,dx
\]

Now, put everything together and take the limits \(\rho \to 0\) and \(R \to \infty\).
\[
\int_{C_R} f(z)\,dz + \int_R^\rho f(z)\,dz + \int_{C_\rho} f(z)\,dz + \int_\rho^R f(z)\,dz = \pi \sqrt{2}
\]
\[
2 \int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2 + 1}\,dx + 0 + 0 = \pi \sqrt{2}
\]
\[
\int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2 + 1}\,dx = \frac{\pi}{\sqrt{2}}
\]

(D) Evaluate the integral
\[
\int_0^\infty \frac{1}{x^a(x + 1)}\,dx
\]
for \(0 < a < 1\).

Let \(f(z) = \frac{z^{-a}}{z + 1}\) with branch 0 < \(\theta < 2\pi\). Let \(C\) be the contour which goes counterclockwise around the circle \(C_R\) of radius \(R\) from \(\theta = 0\) to \(2\pi\), along the underside of the branch cut from \(R\) to \(\rho\), clockwise around the circle \(C_\rho\) of radius \(\rho\) from \(\theta = 2\pi\) down to 0, then along the top of the branch cut from \(\rho\) up to \(R\).

\(f(z)\) has one simple pole inside \(C\) at \(-1\).
\[
\text{Res }_{-1} f(z) = \frac{p(-1)}{q'(-1)} = \frac{e^{-a \log(-1)}}{1} = e^{-ia\pi}
\]
\[
\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_\rho^R f(z)\,dz + \int_{C_\rho} f(z)\,dz + \int_\rho^R f(z)\,dz
\]
\[
= 2\pi ie^{-ia\pi}
\]

Next, \(C_\rho\).
\[
|f(z)| = \frac{|z|^{-a}}{|z + 1|}
\]
\[ \left| \int_{C_\rho} f(z)dz \right| \leq \frac{\rho^{-a}}{1-\rho} \]

\[ \left| \int_{C_\rho} f(z)dz \right| \leq \frac{2\pi\rho^{1-a}}{1-\rho} \]

\[ \lim_{\rho \to 0} \left| \int_{C_\rho} f(z)dz \right| \leq \frac{0}{1-0} = 0 \]

Note, we used the fact that \( a < 1 \) to evaluate the limit. On to \( C_R \).

\[ |f(z)| = \frac{|z|^{-a}}{|z+1|} \leq \frac{R^{-a}}{R-1} \]

\[ \left| \int_{C_R} f(z)dz \right| \leq \frac{2\pi R^{1-a}}{R-1} \]

\[ \lim_{R \to \infty} \left| \int_{C_R} f(z)dz \right| = 0 \]

Here, we used \( a > 0 \) to evaluate the limit. Next, the piece under the branch cut.

\[ \int_{R}^{\rho} f(z)dz = \int_{R}^{\rho} \frac{e^{-a(\ln x + i2\pi)}}{x+1} dx \]

\[ = - \int_{\rho}^{R} \frac{e^{-a \ln x - i a2\pi}}{x+1} dx \]

\[ = \int_{\rho}^{R} \frac{-x^{-a}e^{-ia2\pi}}{x+1} dx \]

Now, put everything together and take the limits \( \rho \to 0 \) and \( R \to \infty \).

\[ \int_{C_\rho} f(z)dz + \int_{R}^{\rho} f(z)dz + \int_{C_\rho} f(z)dz + \int_{\rho}^{R} f(z)dz = 2\pi i e^{-ia\pi} \]

\[ (1-e^{-ia2\pi}) \int_{0}^{\infty} \frac{x^{-a}}{x+1} dx + 0 + 0 = 2\pi i e^{-ia\pi} \]

\[ (e^{ia\pi} - e^{-ia\pi}) \int_{0}^{\infty} \frac{x^{-a}}{x+1} dx + 0 + 0 = 2\pi i \]

\[ (2i \sin a\pi) \int_{0}^{\infty} \frac{x^{-a}}{x+1} dx + 0 + 0 = 2\pi i \]

\[ \int_{0}^{\infty} \frac{x^{-a}}{x+1} dx = \frac{\pi}{2\sin a\pi} \]