Finishing Practice Integral C

(C) Evaluate the integral

$$\int_0^\infty \frac{\ln x}{1+x^4}dx$$

using a quarter circle contour (coming back along the imaginary axis) or the upper half circle.

The only singularity inside the contour is at

$$e^{\frac{i\pi}{4}}$$

We plug that into p/q' to get the residue.

$$\begin{aligned}
\frac{p}{q'} &= \frac{\log z}{4z^3} \\
Res &= \frac{\log e^{\frac{i\pi}{4}}}{4(e^{\frac{i\pi}{4}})^3} \\
&= \frac{i\pi}{16}e^{-\frac{i3\pi}{4}} \\
&= \frac{i\pi}{16}(\cos(-\frac{i3\pi}{4}) + i\sin(-\frac{i3\pi}{4})) \\
&= \frac{i\pi}{16}(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}) \\
\int_C \frac{\log z}{1+z^4}dz &= \frac{-\pi^2}{8}(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}) \\
&= \frac{\pi^2}{8\sqrt{2}}(1+i)
\end{aligned}$$

For the C_- integral, we sub in $z(t) = it$.

$$\begin{aligned}
\int_{C_-} \frac{\log z}{1+z^4}dz &= -\int_\rho^R \frac{\log it}{1+(it)^4}idt \\
&= -i\int_\rho^R \frac{\ln t + i\frac{\pi}{2}}{1+t^4}dt \\
&= -i\int_\rho^R \frac{\ln t}{1+t^4}dt + \frac{\pi}{2}\int_\rho^R \frac{1}{1+t^4}dt
\end{aligned}$$

After you take care of C_ρ and C_R, you're left with

$$\frac{\pi^2}{8\sqrt{2}}(1+i) = \int_0^\infty \frac{\ln x}{1+x^4}dx - i\int_0^\infty \frac{\ln x}{1+x^4}dx + \frac{\pi}{2}\int_0^\infty \frac{1}{1+x^4}dx$$

If you match up real and imaginary parts, the imaginary parts give us the answer we want:

$$i\frac{\pi^2}{8\sqrt{2}} = -i\int_0^\infty \frac{\ln x}{1+x^4}dx$$

$$-\frac{\pi^2}{8\sqrt{2}} = \int_0^\infty \frac{\ln x}{1+x^4} dx$$