1.2 Homework Solutions

(10) Consider the system consisting of all elements of the form $a + b\sqrt{5}$, where a and b are any rational numbers. Show that all 11 axioms of 1.1 are valid (using the usual rules for addition and multiplication), hence this system forms a field.

In the following, we will repeatedly use the fact that the rational numbers are a field, which was stated in class. Note, most of these axioms hold simply because all elements are real and the normal 0 and 1 belong. Only A1, M1 and M5 are not immediately clear.

A1 - $(a + b\sqrt{5}) + (c + d\sqrt{5}) = (a + c) + (b + d)\sqrt{5}$
A2 - Since both $(a+b\sqrt{5})$ and $(c+d\sqrt{5})$ are real, they commute, so $(a+b\sqrt{5})+(c+d\sqrt{5}) = (c + d\sqrt{5}) + (a + b\sqrt{5})$.
A3 - As in A2, all elements are real and so A3 holds here as well.
A4 - Since 0 for the real numbers can be written $0 + 0\sqrt{5}$ (and 0 is rational), it is in the system.
A5 - If a and b are rational, so are $-a$ and $-b$. Therefore $-(a + b\sqrt{5}) = -a + (-b)\sqrt{5}$ is in the system.
M1 - $(a + b\sqrt{5})(c + d\sqrt{5}) = (ac + 5bd) + (ad + bc)\sqrt{5}$. Since $a, b, c, d, 5$ are rational, $ac + 5bd$ and $ad + bc$ are rational and $(ac + 5bd) + (ad + bc)\sqrt{5}$ is in the system.
M2, M3 - All elements are real so M2/M3 hold.
M4 - Since 1 for the real numbers can be written $1 + 0\sqrt{5}$ (and $0, 1$ are rational), it is in the system.
M5 -

$$\begin{aligned}
\frac{1}{a + b\sqrt{5}} &= \frac{1}{a + b\sqrt{5}} \cdot \frac{a - b\sqrt{5}}{a - b\sqrt{5}} \\
&= \frac{a - b\sqrt{5}}{a^2 - 5b^2} \\
&= \frac{a}{a^2 - 5b^2} + \frac{-b}{a^2 - 5b^2}\sqrt{5}
\end{aligned}$$

We see that $\frac{1}{a+b\sqrt{5}}$ is in the system UNLESS $a^2 = 5b^2$, in which case it is undefined. Luckily $a^2 = 5b^2$ has no solutions where both a and b are rational, so the reciprocal is defined for all elements in the system.
D - All elements are real so D holds.