Solutions - 2.1/2.2 Continuous Functions

(9) Show that

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = -6.$$

Let $\epsilon > 0$ be given. Assume $|x - (-3)| < \delta$.

$$\begin{aligned} |\frac{x^2-9}{x+3} - (-6)| &= |x - 3 + 6| \\ &= |x + 3| \\ &< \delta \end{aligned}$$

As long as $|x - (-3)| < \delta = \epsilon$, then $|\frac{x^2-9}{x+3} - (-6)| < \epsilon$.

(12) Show that

$$\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = 6.$$

Let $\epsilon > 0$ be given. Assume $|x - 9| < \delta$.

$$\begin{aligned} |\frac{x-9}{\sqrt{x}-3} - 6| &= |\sqrt{x} + 3 - 6| \\ &= |\sqrt{x} - 3| \\ &= |\frac{x-9}{\sqrt{x}+3}| \quad \text{as long as } \delta \leq 5, \quad x \geq 4 \\ &< \frac{\delta}{\sqrt{4}+3} \\ &= \frac{\delta}{5} \end{aligned}$$

As long as $|x - 9| < \delta = \min\{5, 5\epsilon\}$, then $|\frac{x-9}{\sqrt{x}-3} - 6| < \epsilon$.

(21) For all $x \in \mathbb{R}$, define

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that every value of x, f is not continuous.

Let x be any real number and let $\epsilon = \frac{1}{4}$. Let δ be any positive real number.

Then there exists $N > 0$ large enough such that $\frac{1}{10^N} < \frac{\delta}{2}$. Let r be the decimal expansion of x rounded to $N + 1$ digits. Then $|x - r| < \delta$ and r is rational. We can also see that $|x - (r + \frac{\sqrt{2}}{10^{N+1}})| < \delta$ and $s = r + \frac{\sqrt{2}}{10^{N+1}}$ is irrational. Both r and s are within δ of x. If x is rational, then $|f(x) - f(s)| = |1 - 0| = 1 > \frac{1}{4}$. If x is irrational, then $|f(x) - f(r)| = |0 - 1| = 1 > \frac{1}{4}$. Since δ was an arbitrary positive number, there is no δ which makes the limit work for $\epsilon = \frac{1}{4}$. Therefore, the limit does not exist (for any real x) and f is not continuous (for any real x).