Solutions - 2.1/2.2 Limit Laws

(1) Show that for all numbers a, c, d,

$$\lim_{x \to a} cx + d = ca + d.$$

Let $\epsilon > 0$ be given. Assume $|x - a| < \delta$.

$$\begin{aligned} |(cx + d) - (ca + d)| &= |c(x - a)| \\ &= |c||x - a| \\ &< |c|\delta \end{aligned}$$

If $c = 0$, any δ will do. If $c \neq 0$, as long as $|x - a| < \delta = \frac{\epsilon}{|c|}$, then $|(cx + d) - (ca + d)| < \epsilon$.

(2) Use Theorems 2.2, 2.3, 2.5, 2.6 to show that for all numbers a, c, d, e,

$$\lim_{x \to a} cx^2 + dx + e = ca^2 + da + e.$$

$$\begin{aligned} \lim_{x \to a} cx^2 + dx + e &= \lim_{x \to a} cx^2 + \lim_{x \to a} dx + \lim_{x \to a} e \qquad (2.5) \\ &= (\lim_{x \to a} c)(\lim_{x \to a} x)(\lim_{x \to a} x) + (\lim_{x \to a} d)(\lim_{x \to a} x) + \lim_{x \to a} e \qquad (2.6) \\ &= (c)(\lim_{x \to a} x)(\lim_{x \to a} x) + (d)(\lim_{x \to a} x) + e \qquad (2.2) \\ &< (c)(a)(a) + (d)(a) + e \qquad (2.3) \\ &= ca^2 + da + e \end{aligned}$$

Prove Corollary 2 part 1: The function $f(x) = \frac{1}{x}$ is continuous for every value of $x \neq 0$.

Let a be any real number other than 0 and let $\epsilon > 0$ be given. Assume $|x - a| < \delta$.

$$\begin{aligned} |\frac{1}{x} - \frac{1}{a}| &= |\frac{a - x}{ax}| \\ &= \frac{|x - a|}{|a||x|} \\ &< \delta \frac{1}{|a|} \frac{1}{|x|} \end{aligned}$$

If we ensure that $\delta < \frac{|a|}{2}$, then $\frac{a}{2} < x < \frac{3a}{2}$ and $\frac{1}{|x|} < \frac{2}{|a|}$. So

$$|\frac{1}{x} - \frac{1}{a}| < \delta \frac{1}{|a|} \frac{1}{|x|} < \delta \frac{1}{|a|} \frac{2}{|a|}$$

Then as long as $|x - a| < \delta = \min\{\frac{|a|}{2}, \frac{\epsilon |a|^2}{2}\}$, then $|\frac{1}{x} - \frac{1}{a}| < \epsilon$.

(13) Prove Th 2.8 (Limit of a Quotient): Suppose that

$$\lim_{x \to a} f(x) = L \quad , \quad \lim_{x \to a} g(x) = M$$

If $M \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$$

This can be proved by applying Theorems 2.6, 2.7 and Corollary 2. By (2.6),

$$\lim_{x \to a} \frac{f(x)}{g(x)} = (\lim_{x \to a} f(x))(\lim_{x \to a} \frac{1}{g(x)})$$

By (2.7) and Cor 2 (which says $1/x$ is continuous),

$$(\lim_{x \to a} f(x))(\lim_{x \to a} \frac{1}{g(x)}) = (\lim_{x \to a} f(x))(\frac{1}{\lim_{x \to a} g(x)})$$

Then we just plug in our assumptions.

$$(\lim_{x \to a} f(x))(\frac{1}{\lim_{x \to a} g(x)}) = L(\frac{1}{M}) = \frac{L}{M}$$