Solutions - 2.3/2.4 - One Sided and Infinite Limits

(3) For the function $f(x) = \frac{|x|}{x}$, find the limit at $x = 0$ from the left and right, and determine if the function is continuous at 0.

Let $\epsilon > 0$ be given. Assume $0 < x < \delta$.

$$\begin{aligned} |\frac{|x|}{x} - 1| &= |\frac{x}{x} - 1| \\ &= |1 - 1| \\ &= 0 \end{aligned}$$

Works for any δ. So $\lim_{x\to 0^+} \frac{|x|}{x} = 1$.

Now assume $-\delta < x < 0$.

$$\begin{aligned} |\frac{|x|}{x} - (-1)| &= |\frac{-x}{x} + 1| \\ &= |-1 + 1| \\ &= 0 \end{aligned}$$

Works for any δ. So $\lim_{x\to 0^-} \frac{|x|}{x} = -1$. Since these limits are different, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist and $f(x)$ is not continuous at $x = 0$.

(6) Evaluate the limit

$$\lim_{x\to 1^+} \frac{(x-1)}{\sqrt{x^2-1}}$$

Let $\epsilon > 0$ be given. Assume $0 < x - 1 < \delta$.

$$\begin{aligned} |\frac{(x-1)}{\sqrt{x^2-1}} - 0| &= |\frac{(x-1)}{\sqrt{(x-1)(x+1)}}| \\ &= |\frac{\sqrt{x-1}}{\sqrt{x+1}}| \\ &< \frac{\sqrt{\delta}}{1+1} \end{aligned}$$

As long as $0 < x - 1 < \delta = 4\epsilon^2$, then $|\frac{(x-1)}{\sqrt{x^2-1}} - 0| < \epsilon$. Therefore, $\lim_{x\to 1^+} \frac{(x-1)}{\sqrt{x^2-1}} = 0$.

(8) Evaluate the limit

$$\lim_{x\to +\infty} \frac{(x^2+1)}{x^{\frac{3}{2}}}$$

Let $M > 0$ be given. Assume $x > N$.

$$
\begin{aligned}
|\frac{(x^2+1)}{x^{\frac{3}{2}}}| &= |\frac{x^2}{x^{\frac{3}{2}}} + \frac{1}{x^{\frac{3}{2}}}| \\
&= |x^{\frac{1}{2}} + \frac{1}{x^{\frac{3}{2}}}| \\
&> |x^{\frac{1}{2}}| \\
&> N^{\frac{1}{2}}
\end{aligned}
$$

As long as $x > N = M^2$, then $|\frac{(x^2+1)}{x^{\frac{3}{2}}}| > M$. Therefore, $\lim_{x\to+\infty} \frac{(x^2+1)}{x^{\frac{3}{2}}} = +\infty$.

(11) Prove Theorem 2.14:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Let $\epsilon > 0$ be given. Assume $|x| > M$.

$$
\begin{aligned}
|\frac{1}{x} - 0| &= \frac{1}{|x|} \\
&< \frac{1}{M}
\end{aligned}
$$

As long as $|x| > M = \frac{1}{epsilon}$, then $|\frac{1}{x}| < \epsilon$.