Problems - 2.5 - Limits of Sequences

(1) Given the function $f(x) = x \cos x$.

(a) Find a sequence of numbers $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} f(x_n) = 0$$

Choose $x_n = (n + \frac{1}{2})\pi$. $\lim_{n\to\infty} x_n = \infty$ since each $x_n > n$ so Th 2.21 applies. Since $\cos(n + \frac{1}{2})\pi =$ for any natural number n, $\lim_{n\to\infty} f(x_n) = 0$.

(b) Find a sequence of numbers $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} f(x_n) = +\infty$$

Choose $x_n = n2\pi$. $\lim_{n\to\infty} x_n = \infty$ since each $x_n > n$ so Th 2.21 applies. Since $\cos n2\pi = 1$ for any natural number n, $f(x_n) = n2\pi > n$ and $\lim_{n\to\infty} f(x_n) = \infty$, again by 2.21.

(c) Find a sequence of numbers $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} f(x_n) = -\infty$$

Choose $x_n = (2n + 1)\pi$. $\lim_{n\to\infty} x_n = \infty$ since each $x_n > n$ so Th 2.21 applies. Since $\cos(2n+1)\pi = -1$ for any natural number n, $-f(x_n) = (2n+1)\pi > n$ and $\lim_{n\to\infty} -f(x_n) = \infty$, again by 2.21. Therefore, for any $M > 0$, there is a number N such that $n > N$ implies $-f(x_n) > M$. It follows that $f(x_n) < -M$, so $\lim_{n\to\infty} f(x_n) = -\infty$.

(6) If $-1 < a < 1$, show that $\lim_{n\to\infty} a^n = 0$.

If $a = 0$, $a^n = 0$ for all n, so the result is clear. Suppose $a \neq 0$.

Let $\epsilon > 0$ be given. We to find N such that $n > N$ implies $|a^n - 0| < \epsilon$. Let's proceed by contradiction. Suppose $|a^n| > \epsilon$ for all $n \in \mathbb{N}$. Then $\frac{1}{|a|^n} < \frac{1}{\epsilon}$ for all n. Since $|a| < 1$, $\frac{1}{|a|} > 1$ and $\frac{1}{|a|^{n+1}} > \frac{1}{|a|^n}$. By Axiom C, there is a number $L \le \frac{1}{\epsilon}$ such that $\lim_{n\to\infty} \frac{1}{|a|^n} = L$ and $\frac{1}{|a|^n} \le L$ for all n.

The limit implies there exists N such that $n > N$ implies $|\frac{1}{|a|^n} - L| < (1 - |a|)L$. Then

$$\begin{aligned} |\frac{1}{|a|^{N+1}} - L| &< (1-|a|)L \\ L - \frac{1}{|a|^{N+1}} &< L - |a|L \\ |a|L &< \frac{1}{|a|^{N+1}} \\ L &< \frac{1}{|a|^{N+2}} \end{aligned}$$

This last line is a contraction to the consequence of Axiom C ($\frac{1}{|a|^n} \le L$). As a result, we must have $|a^n - 0| < \epsilon$, and the limit is verified.