Problems - 2.2-2.4 Extras

(1) Verify the limit

$$\lim_{x\to 0^+} \sqrt{x} = 0$$

Let $\epsilon > 0$ be given. Assume $0 < x - 0 < \delta$.

$$\begin{aligned} |\sqrt{x} - 0| &= \sqrt{x} \\ &< \sqrt{\delta} \end{aligned}$$

Then as long as $0 < x - 0 < \delta = \epsilon^2$, $|\sqrt{x} - 0| < \epsilon$.

(2) Verify the limit

$$\lim_{x\to 4} \frac{x}{x-4} = \infty$$

Let $M > 0$ be given. Assume $|x - 4| < \delta$. Note, as long as $\delta \leq 1$, $3 < x < 5$.

$$\begin{aligned} |\frac{x}{x-4}| &= \frac{|x|}{|x-4|} \\ &> \frac{3}{\delta} \end{aligned}$$

Then as long as $|x - 4| < \delta = \min\{1, \frac{3}{M}\}$, $|\frac{x}{x-4}| > M$.

(3) Prove the following, or find a counterexample: Suppose f and g are defined for $x > 0$ and $\lim_{x\to 0^+} f(x) = 0$. Then

$$\lim_{x\to 0^+} f(x)g(x) = 0$$

This statement is false. If $f(x) = x$, we know $\lim_{x\to 0} f(x) = 0$ and therefore $\lim_{x\to 0^+} f(x) = 0$. Let $g(x) = \frac{1}{x}$, which is also defined for $x > 0$. Then

$$\lim_{x\to 0^+} f(x)g(x) = \lim_{x\to 0^+} 1 = 1 \neq 0.$$

(4) Prove the following, or find a counterexample: Suppose f and g are defined for all real numbers, $\lim_{x\to +\infty} f(x) = L > 0$ and $\lim_{x\to +\infty} g(x) = +\infty$. Then

$$\lim_{x\to +\infty} f(x)g(x) = +\infty$$

This statement is true. Let $M > 0$ be given. As long as we make sure $f(x)$ is not near 0, we can ensure $g(x)$ is large enough to make $f(x)g(x) > M$.

Since $\lim_{x\to +\infty} f(x) = L$, then (using $\epsilon = \frac{L}{2}$*) there is a number $N_f > 0$ such that $x > N_f$ implies $\frac{L}{2} < f(x) < \frac{3L}{2}$.

Since $\lim_{x\to +\infty} g(x) = +\infty$, there is a number $N_g > 0$ such that $x > N_g$ implies $g(x) > \frac{M}{(L/2)}$.

Then $x > \max\{N_f, N_g\}$ implies

$$f(x)g(x) > \frac{L}{2}\frac{M}{(L/2)} = M$$

* Note - We must choose ϵ to ensure $f(x) \neq 0$, otherwise $f(x) - \epsilon$ could be 0 or negative. In either case, the last line concerning $f(x)g(x)$ wouldn't work.