Solutions - 3.2

(2) Find $\inf S$ and $\sup S$, and state whether or not these are contained in S.

$$S = \{x : x^2 - 3 < 0\}$$

$\inf S = \sqrt{3}$ and $\sup S = -\sqrt{3}$. As a quick check, let ϵ be a real number such that $0 < \epsilon < 2\sqrt{3}$. Then $\sqrt{3} - \epsilon$ is not an upper bound, since

$$(\sqrt{3} - \frac{\epsilon}{2})^2 - 3 = -\epsilon\sqrt{3} + \frac{\epsilon^2}{4} = -\frac{\epsilon}{2}(2\sqrt{3} - \frac{\epsilon}{2}) < -\frac{\epsilon}{2}(\epsilon - \frac{\epsilon}{2}) < 0$$

Similarly, $-\sqrt{3} + \epsilon$ is not a lower bound. Note, neither $\pm\sqrt{3}$ are in S as neither satisfies the inequality.

(4) Find $\inf S$ and $\sup S$, and state whether or not these are contained in S.

$$S = \{x : x = \frac{y}{y+1}, \quad y \geq 0\}$$

Since $f(y) = \frac{y}{y+1} = 1 - \frac{1}{y+1}$, we can see that $f(y)$ is increasing for $y \geq 0$. So we have $\inf S = \lim_{x \to 0^+} f(y) = f(0) = 0$ (since f is a rational function and the denominator is not 0 at $y = 0$, f is continuous at 0 by Theorems 2.2, 2.3, 2.5 and 2.8.). We also have

$$\sup S = \lim_{x \to +\infty} f(y) = \lim_{x \to +\infty} \frac{1}{1 + \frac{1}{y}} = \frac{\lim_{x \to +\infty} 1}{\lim_{x \to +\infty} 1 + \lim_{x \to +\infty} \frac{1}{y}} = \frac{1}{1+0} = 1$$

(10) Suppose $B_1 = \sup S_1$, $B_2 = \sup S_2$ and $S_1 \subset S_2$. Show that $B_1 \leq B_2$.

Suppose $B < B_1$. Since $B_1 = \sup S_1$, we can find $x \in S_1$ such that $B < x < B_1$. But since $S_1 \subset S_2$, x is also in S_2. Therefore, B cannot be an upper bound for S_2. Since B was an arbitrary number less than B_1, any upper bound for S_2, including B_2, must be greater or equal B_1.

(11) Suppose the S_1, S_2, S_3 are sets in $\mathbb{R}$ and $S = S_1 \cup S_2 \cup S_3$. Show that $\inf S = \min(\inf S_1, \inf S_2, \inf S_3)$.

Let $b = \min(\inf S_1, \inf S_2, \inf S_3)$. Need to check that b is a lower bound, then that b is the greatest lower bound.

Suppose $x \in S$. Then $x \in S_i$ for at least one of $i = 1, 2, 3$. This implies $x \geq \inf S_i \geq \min(\inf S_1, \inf S_2, \inf S_3) = b$. So b is a lower bound on S.

Now suppose $a > b$. Then $a > \inf S_j$ for at least one of $j = 1, 2, 3$. This implies we can find $y \in S_j$ such that $\inf S_j < y < a$. Since y is also in S, a is not a lower bound on S_j or S. Since a could be any number greater than b, $b = \inf S$.