Solutions - 3.5

(5) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $S = [1, \infty)$.

Let $\epsilon > 0$ be given. Assume $|x_1 - x_2| < \delta$ and $x_1, x_2 \in S$.

$$\begin{aligned}
|\frac{1}{x_1} - \frac{1}{x_2}| &= \frac{|x_2 - x_1|}{|x_1 x_2|} \\
&< \frac{\delta}{|x_1||x_2|} \\
&\leq \frac{\delta}{(1)(1)} \qquad \text{since } x_1, x_2 \geq 1 \\
&= \delta
\end{aligned}$$

So as long as $|x_1 - x_2| < \delta = \epsilon$ and $x_1, x_2 \in S$, we have $|\frac{1}{x_1} - \frac{1}{x_2}| < \epsilon$.

(9) Show using ϵ and δ (not Th. 3.13) that $f(x) = \sqrt{x}$ is uniformly continuous on $I = [0, 1]$.

Let $\epsilon > 0$ be given. Assume $|x_1 - x_2| < \delta$ and $x_1, x_2 \in I$.

$$\begin{aligned}
|\sqrt{x_1} - \sqrt{x_2}| &= \frac{|\sqrt{x_1} - \sqrt{x_2}||\sqrt{x_1} + \sqrt{x_2}|}{|\sqrt{x_1} + \sqrt{x_2}|} \\
&= \frac{|x_1 - x_2|}{|\sqrt{x_1} + \sqrt{x_2}|} \\
&< \frac{\delta}{|\sqrt{x_1} + \sqrt{x_2}|}
\end{aligned}$$

Uh-oh. That denominator could be really small, so this isn't going to work. But x_1 and x_2 very close to 0 is not hard to deal with another way. If $0 \leq x_1 < \epsilon^2$, then $\sqrt{x_1} < \epsilon$. Same for x_2. Their difference is even smaller, so $|x_1 - x_2| < \epsilon^2$ and $|\sqrt{x_1} - \sqrt{x_2}| < \epsilon$ for these small x's. If at least one of the x's is bigger or equal to ϵ^2 (so $\sqrt{x_i} > \epsilon$), then we can go back to the old plan.

$$\begin{aligned}
|\sqrt{x_1} - \sqrt{x_2}| &< \frac{\delta}{|\sqrt{x_1} + \sqrt{x_2}|} \\
&\leq \frac{\delta}{\epsilon}
\end{aligned}$$

So as long as $|x_1 - x_2| < \delta = \epsilon^2$ and $x_1, x_2 \in I$, we have $|\sqrt{x_1} - \sqrt{x_2}| < \epsilon$.

(10) Suppose the f is uniformly continuous on the half-open interval $I = (0, 1]$. Is it true that f must be bounded on I? If so, prove it. If not, give a counter-example.

This is true. Pick an $\epsilon > 0$. Since f is uniformly continuous, there exists a δ such that $|x_1 - x_2| < \delta$ and $x_1, x_2 \in (0, 1]$ implies $|f(x_1) - f(x_2)| < \epsilon$.

Pick a number c between 0 and δ (such as $\frac{\delta}{2}$). Then $|x_1 - x_2| = c$ and $x_1, x_2 \in (0,1]$ implies $|f(x_1) - f(x_2)| < \epsilon$. Starting from $f(1)$, if we step left by c, f can only go up or down by ϵ. Each step can only add or subtract ϵ. So

$$\begin{aligned}
f(1) - \epsilon &\leq f(1-c) \leq & f(1) + \epsilon \\
f(1) - 2\epsilon &\leq f(1-2c) \leq & f(1) + 2\epsilon \\
f(1) - 3\epsilon &\leq f(1-3c) \leq & f(1) + 3\epsilon \\
f(1) - k\epsilon &\leq f(1-kc) \leq & f(1) + k\epsilon
\end{aligned}$$

Every x in $(0,1]$ can be reach by going left from $x = 1$ in $\frac{1}{c}$ steps of size c or less. Since you took $\frac{1}{c}$ steps or fewer, f can only change by at most $\epsilon\frac{1}{c}$. Therefore, $f(1) - \epsilon\frac{1}{c} \leq f(x) \leq f(1) + \epsilon\frac{1}{c}$ for all $x \in (0,1]$.