Solutions - 3.6

(4) Decide if the given sequence is a Cauchy sequence. If it is, prove it. If it is not, a subsequence which is a Cauchy sequence.

$$x_n = 1 + (-1)^n + \frac{1}{n}$$

Since $\frac{1}{n}$ goes to 0, this sequences essentially alternates between 0 and 2, and hence does not converge. The even terms $2 + \frac{1}{2n}$ converge to 2, so that subsequence is Cauchy (works for odd terms too).

(7) Suppose that $s_n = \sum_{j=1}^{n} u_j$ and $S_n = \sum_{j=1}^{n} |u_j|$ for $n \in \mathbb{N}$. Assume $\lim_{n\to\infty} S_n = S$. Show that s_n converges as n goes to infinity.

Let $\epsilon > 0$ be given. Since S_n converges, it is Cauchy. Therefore, we can find $N > 0$ so that if $n, m > N$, we have

$$|S_n - S_m| < \epsilon$$

Assume, without loss of generality, that $n \geq m$. Then

$$|S_n - S_m| = |u_{m+1}| + |u_{m+2}| + \cdots + |u_n| < \epsilon.$$

Meanwhile, we can do something similar for s_n, then relate the two sequences by the Triangle Inequality.

$$\begin{aligned} |s_n - s_m| &= |u_{m+1} + u_{m+2} + \cdots + u_n| \\ &\leq |u_{m+1}| + \cdots + |u_n| \qquad \text{by Tri. Ineq.} \\ &< \epsilon \end{aligned}$$

So s_n is Cauchy too, and therefore converges.

(9) Suppose that f has a domain which contains interval $I = (a, b)$, and suppose that $\lim_{x\to b^-} f(x) = L$. Prove that for each $\epsilon > 0$, there is a $\delta > 0$ such that $b - \delta < x, y < b$ implies $|f(x) - f(y)| < \epsilon$. (Note: You are proving one direction of Th. 3.14 for one-sided limits.)

Let $\epsilon > 0$ be given. Since $\lim_{x\to b^-} f(x) = L$, we can find $\delta > 0$ so that if $b - \delta < x < b$, then $|f(x) - L| < \frac{\epsilon}{2}$ (we use $\frac{\epsilon}{2}$ since we are aiming for $< \epsilon$ at the end). Then if both x and y are between $b - \delta$ and b, we have

$$|f(x) - f(y)| = |f(x) - L + L - f(y)| \leq |f(x) - L| + |L - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$