Solutions - 3.7

(13) (a) Show that the family F of all intervals of the form $I_n = (\frac{1}{n+2}, \frac{1}{n})$ for $n \in \mathbb{N}$ covers the interval $J = (0, 1)$.

For this problem, a picture with the first few intervals showing the repeating pattern is sufficient. If you are set on doing a rigorous proof, it might look something like the following:

Let c be a number in $J = (0, 1)$. Since the sequence $\frac{1}{n}$ for $n \in \mathbb{N}$ is decreasing, starts at 1 and converges to 0, we can find a positive integer k such that $\frac{1}{k+1} \leq c < \frac{1}{k}$. Then $c \in (\frac{1}{k+2}, \frac{1}{k})$. Since c was arbitrary, F covers J.

(b) Show that no subfamily (finite or infinite) of F covers J.

Any proper subfamily would have to remove at least one interval I_n from F. If I_n is removed, then the number $\frac{1}{n+1}$ is not contained in any of the remaining intervals. Therefore no proper subfamily of F covers $(0, 1)$.

(15) Let F_1 be the family of all intervals $I_n = (\frac{1}{2^n}, 2)$ for $n \in \mathbb{N}$. Show the F_1 covers that interval $J = (0, 1)$. Does any finite subfamily of F_1 cover J? Prove your answer.

As in 13(a), a simple argument would be sufficient. The careful argument would say that for any $c \in J = (0, 1)$, we can find k such that $\frac{1}{2^k} < c$. Then $c \in (\frac{1}{2^k}, 2)$. So F_1 covers J.

If we take a finite subfamily of F_1, then one of the intervals in the subfamily has the highest subscript; in other words, it was latest in the original list, and consequently is the biggest interval in the subfamily since the intervals are nested. Suppose that biggest interval is $I_m = (\frac{1}{2^m}, 2)$. Then any points between 0 and $\frac{1}{2^m}$ are not in I_m, or any of the subfamily members (since I_m was biggest). Therefore, no finite subfamily can cover J.