Solutions - 3.7 Cantor Set

(A) Use the geometric series formula (from Calc 2) to show that

$$\sum_{n=1}^{\infty} \frac{2}{3^{2n}} = \frac{1}{4}$$

Conclude that $\frac{1}{4}$ is in the Cantor Set.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2}{3^{2n}} &= \frac{2}{9}\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \\ &= \frac{2}{9}\frac{1}{1-\frac{1}{9}} \\ &= \frac{2}{9}\frac{9}{8} \\ &= \frac{1}{4}\end{aligned}$$

Since the base 3 expansion of $\frac{1}{4}$ has only 0 and 2 coefficients, $\frac{1}{4}$ is in the Cantor Set.

(B) Let c be any point in the Cantor Set, and let I be an open interval containing c. Show that I also contains points that are not in the Cantor Set.

Method 1: Let δ be the distance from c to the nearest edge of I. Then for some n, $\frac{1}{3^n} < \delta$. Let x be the point which has the same base 3 expansion as c, expect that the coefficient of $\frac{1}{3^n}$ is 1. Then x is not in the Cantor Set (since it has a 1 in the base 3 expansion) but x is contained in I.

Method 2: Let δ be the distance from c to the nearest edge of I. Then for some n, $\frac{1}{3^n} < \delta$. At the n-step of removing middle thirds, c (and the rest of the Cantor Set) is in closed intervals of width $\frac{1}{3^n}$. The closed interval containing c is entirely contained in I, and just outside that closed interval (but still in I) we have stuff which was just removed (and so not in the Cantor Set).

(C) Let F be a family of opens intervals which covers the Cantor Set. Show that a finite subfamily of F covers the Cantor Set.

Proceed by contradiction, very similar to the Heine-Borel proof. Suppose that there is no finite subfamily covering the Cantor Set, which is contained in $I_0 = [0, 1]$.

Remove the middle third of $[0, 1]$. Then we need an infinite subfamily to cover at least one of the Cantor Set contained in the bottom third or the top third. Let I_1 be the third which requires an infinite number. Note, I_1 has width $\frac{1}{3}$.

Repeat. At each step, remove the middle third from I_k (which has width $\frac{1}{3^k}$). Either the piece of the Cantor Set in the bottom piece needs an infinite subfamily, or the piece in the top piece does. Let I_{k+1} be the interval width $\frac{1}{3^{k+1}}$ which needs an infinite subfamily.

The Nested Intervals theorem applies, and the intervals I_k squeeze down to a single point, x_0. Since x_0 was in each I_k, it is in the Cantor Set. Since F covers the Cantor Set, there is an interval $J \in F$ containing x_0. But then there will be an I_k which is small enough that $x_0 \in I_k \subset J$ (choose k so that $\frac{1}{3^k}$ is less than the distance from x_0 to the nearest edge of J). This is a contradiction, since J (a finite subfamily) covers all of I_k, so certainly covers the part of the Cantor Set inside I_k.