Solutions - 4.1 - Theorem Proofs

Prove the following theorems. You may refer to other theorems as long as they have lower numbers.

(Thm 4.2) Suppose f is defined on open interval I, c is a real number and $g(x) = cf(x)$. If $f'(x)$ exists, then $g'(x)$ exists and $g'(x) = cf'(x)$.

(Proof) -

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0} c\frac{f(x+h)-f(x)}{h} \\
&= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

(Thm 4.6) Suppose u and v are defined on open interval I, $v \neq 0$ on I and $f(x) = \frac{u(x)}{v(x)}$. If u' and v' exist, then $f'(x)$ exists and $f'(x) = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$.

(Proof) -

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x)-u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{u(x+h)-u(x)}{hv(x+h)} - \lim_{h\to 0} \frac{u(x)v(x+h)-u(x)v(x)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \lim_{h\to 0} \frac{1}{v(x+h)} - \frac{u(x)}{v(x)} \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \lim_{h\to 0} \frac{1}{v(x+h)} \\
&= u'(x)\frac{1}{v(x)} - \frac{u(x)}{v(x)}v'(x)\frac{1}{v(x)} \\
&= \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}
\end{aligned}$$

Note that in the second-last line, we used the fact that v' exists implies v is continuous.

(Thm 4.10) Suppose f is continuous on open interval I, f takes on its maximum at x_0, and x_0 is an interior point of I (that is, x_0 is not one of the endpoints of I). If $f'(x_0)$ exists, then $f'(x_0) = 0$.

Since $f'(x_0)$ exists, $f'(x_0) = \lim h \to 0^+ \frac{f(x_0+h)-f(x_0)}{h} = \lim h \to 0^- \frac{f(x_0+h)-f(x_0)}{h}$. But since $f(x_0)$ is the maximum of $f(x)$, $f(x_0) \geq f(x_0 + h)$, which means $f(x_0 + h) - f(x_0) \leq 0$. So we have both

$$
\begin{aligned}
f'(x_0) = \lim h \to 0^+ \frac{f(x_0 + h) - f(x_0)}{h} &\leq 0 \\
f'(x_0) = \lim h \to 0^- \frac{f(x_0 + h) - f(x_0)}{h} &\geq 0 \quad \text{since } h \text{ is negative)}
\end{aligned}
$$

Since $f'(x_0)$ is both greater or equal and less than or equal to 0, it must be 0.