Solutions - 4.1 - continued

(8) In class we prove Theorem 4.7 (Power Rule) for positive integers. Use that result, along with the Quotient Rule (Thm. 4.6) to check the Power Rule for negative integers.

Suppose n is a positive integer.

$$\begin{aligned} f(x) &= x^{-n} \\ &= \frac{1}{x^n} \\ f'(x) &= \frac{0(x^n) - 1(nx^{n-1})}{x^{2n}} \\ &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{-n-1} \end{aligned}$$

(9a) Suppose $f(x) = x^3$ and $x_0 = 2$. Check that the $\eta(h)$ function from Theorem 4.8 is

$$\eta(h) = 6h + h^2$$

Using the formula from the proof of Thm. 4.8, if $h \neq 0$,

$$\begin{aligned} \eta(h) &= \frac{f(2+h) - f(2)}{h} - f'(2) \\ &= \frac{(2+h)^3 - 2^3}{h} - 3(2)^2 \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} - 12 \\ &= 12 + 6h + h^2 - 12 \\ &= 6h + h^2 \end{aligned}$$

When $h = 0$, $6h + h^2 = 0$ so that matches the η definition too.

(21/22) Evaluate the following two limits (use derivative formulas you know from Calc I, along with l'Hopital's).

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \qquad \lim_{x\to +\infty} \frac{x^3}{e^x}$$

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{2\sec x \sec x \tan x}{6x} \\ &= \lim_{x\to 0} \frac{\sin x}{3x\cos^3 x} \\ &= \lim_{x\to 0} \frac{\cos x}{3\cos^3 x + 3x3\cos^2 x(-\sin x)} \\ &= \frac{1}{3} \end{aligned}$$

$$
\begin{aligned}
\lim_{x\to+\infty} \frac{x^3}{e^x} &= \lim_{x\to+\infty} \frac{3x^2}{e^x} \\
&= \lim_{x\to+\infty} \frac{6x}{e^x} \\
&= \lim_{x\to+\infty} \frac{6}{e^x} \\
&= 0
\end{aligned}
$$