Solutions - 6.1

(1) Show that $\mathbb{R}^n$ with metric

$$d(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

is a metric space.

- d defined: Since the vectors have a finite number of terms, the max is always defined.
- $d(x,y) \geq 0$: Thanks to the absolute values, we are comparing positive terms, so d is always ≥ 0. The max is only $= 0$ if all terms in the max are 0; in other words, x and y must be exactly the same.
- $d(x,y) = d(y,x)$: Since $|x_i - y_i| = |y_i - x_i|$ always, this property is clear.
- Triangle Inequality:

$$\begin{aligned} d(x,z) &= \max_{1 \leq i \leq n} |x_i - z_i| \\ &= \max_{1 \leq i \leq n} |x_i - y_i + y_i - z_i| \\ &\leq \max_{1 \leq i \leq n} (|x_i - y_i| + |y_i - z_i|) \\ &\leq \max_{1 \leq i \leq n} |x_i - y_i| + \max_{1 \leq i \leq n} |y_i - z_i| \\ &= d(x,y) + d(y,z) \end{aligned}$$

(15) Let S be the set of all absolutely convergent series. Show that S with metric

$$d(x,y) = \sum_{i=1}^{\infty} |x_i - y_i|$$

is a metric space.

- d defined:

$$\sum_{i=1}^{\infty} |x_i - y_i| \; leq \sum_{i=1}^{\infty} (|x_i| + |y_i|) = \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i|$$

  Since the x and y series are absolutely convergent, the last sums above converge so the $d(x,y)$ series converges.
- $d(x,y) \geq 0$: Thanks to the absolute values, we are summing positive terms, so d is always ≥ 0. The series is only $= 0$ if all terms are 0 since we can't have negative terms; in other words, x and y must be exactly the same.
- $d(x,y) = d(y,x)$: Since $|x_i - y_i| = |y_i - x_i|$ always, this property is clear.

- Triangle Inequality:

$$\begin{aligned}
d(x,z) &= \sum_{n=1}^{\infty} |x_i - z_i| \\
&= \sum_{n=1}^{\infty} |x_i - y_i + y_i - z_i| \\
&\leq \sum_{n=1}^{\infty} (|x_i - y_i| + |y_i - z_i|) \\
&= \sum_{n=1}^{\infty} |x_i - y_i| + \sum_{n=1}^{\infty} |y_i - z_i| \\
&= d(x,y) + d(y,z)
\end{aligned}$$